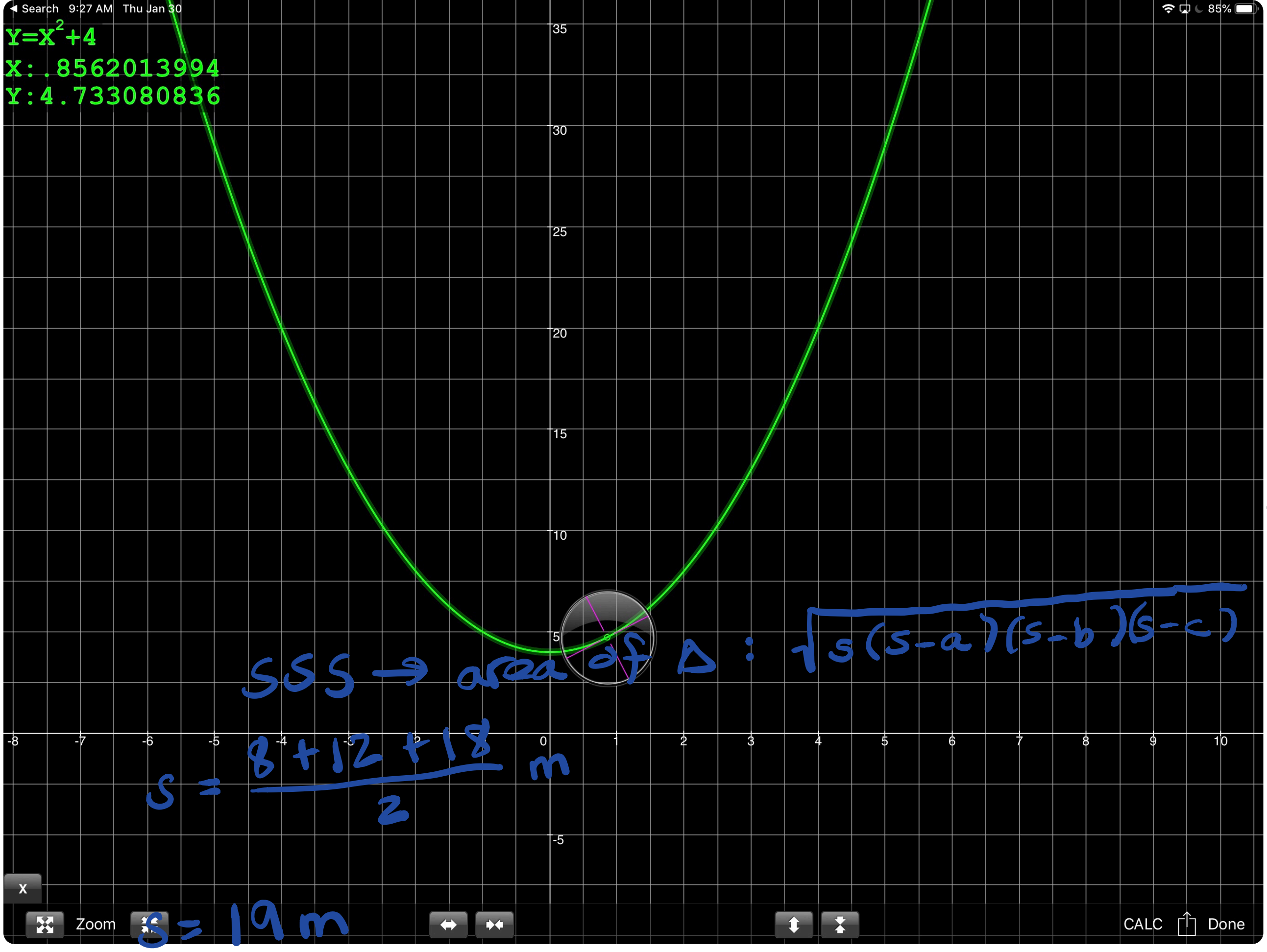


8 m, 12 m,
Feel free

SSS → area of Δ : $\sqrt{s(s-a)(s-b)(s-c)}$, $s = \frac{a+b+c}{2}$

$$s = \frac{8+12+18}{2} \text{ m}$$

$$s = 19 \text{ m}$$

$$A_{\Delta} = \sqrt{19(19-8)(19-12)(19-18)} \text{ m}^2$$

$$= \sqrt{19(11)(7)(1)}$$

$$\approx \boxed{38.2 \text{ m}^2}$$

Calculus is the mathematics of change. What does this mean?? Well, think about traveling on highway 5 when there's no traffic (if that's even possible☺). Suppose you travel from MiraCosta College to go to the races in Del Mar. It takes you 40 minutes to travel and also park at the racetrack. The total distance you traveled is 19 miles. From your precalculus knowledge, are you able to find this rate with the given information? Yes !! . If yes, find the rate.

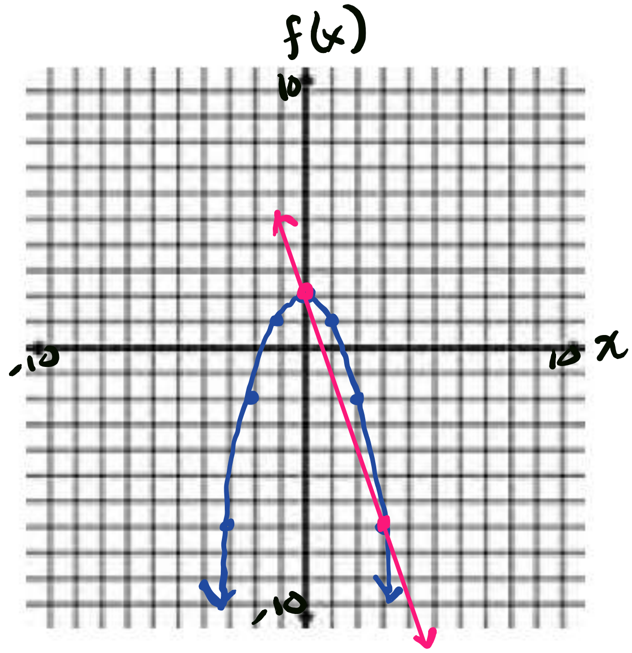
$$d = rt$$
$$19 = r(40)$$
$$\frac{19}{40} = r$$

$$r = \frac{19\,mi}{\overset{}{\cancel{40}}_{2}\,\cancel{min}} \cdot \frac{\overset{3}{\cancel{60}}\,\cancel{min}}{1\,hr}$$

$$\boxed{r = 28.5\,mph}$$

What type of rate of change did you find? The average rate of change, assuming that you traveled at a constant speed. Is this realistic? Nope. Suppose you passed by the highway patrol while you were on the freeway, and he used his radar gun to find out your rate? This would measure your rate at the exact moment he got the reading. This type of rate of change is instantaneous. This leads us to the tangent line problem.

Example 1: Find the equation of the line $\overbrace{\text{secant}}^{\text{to cut}}$ to graph of $f(x) = -x^2 + 2$ at $x = 0$ and $x = 3$. $\rightarrow \Delta x = 3 - 0 = 3$



$$f(x) = -x^2 + 2$$

$$m_{sec} = \frac{f(x+\Delta x) - f(x)}{(x+\Delta x) - x}$$

$$= \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \frac{f(x+3) - f(x)}{3}$$

$$f(0) = -0^2 + 2 = 2$$

$$f(0+3) = f(3) = -3^2 + 2 = -7$$

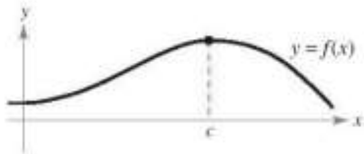
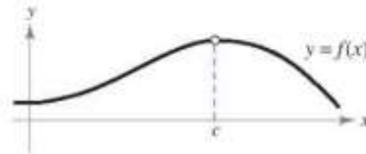
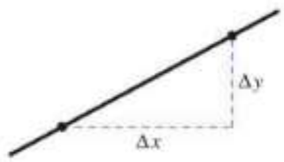
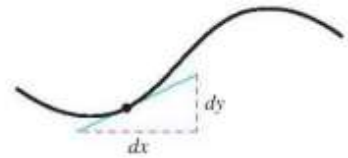
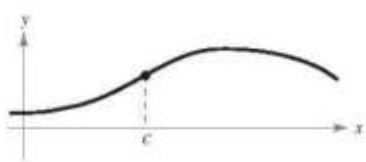
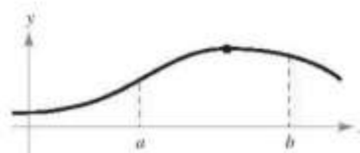
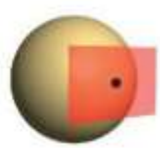
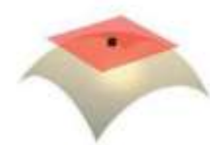
$$m_{sec} = \frac{-7-2}{3} = -3$$

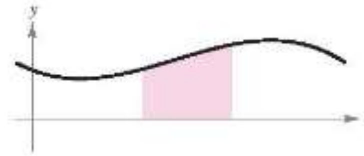
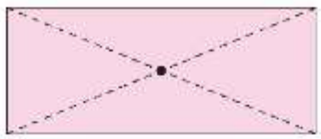
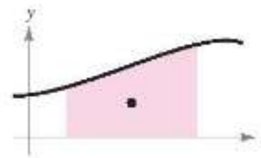
$$\boxed{\begin{aligned} y - 2 &= 3(x-0) \\ y - 2 &= 3x \end{aligned}}$$

The main difference between precalculus and calculus is the addition of the

limit process to your calculations, and thinking of

instantaneous change versus average change.

Without Calculus		With Differential Calculus	
Value of $f(x)$ when $x = c$		Limit of $f(x)$ as x approaches c	
Slope of a line		Slope of a curve	
Secant line to a curve		Tangent line to a curve	
Average rate of change between $t = a$ and $t = b$		Instantaneous rate of change at $t = c$	
Curvature of a circle		Curvature of a curve	
Height of a curve when $x = c$		Maximum height of a curve on an interval	
Tangent plane to a sphere		Tangent plane to a surface	
Direction of motion along a line		Direction of motion along a curve	

Without Calculus		With Integral Calculus	
Area of a rectangle		Area under a curve	
Work done by a constant force		Work done by a variable force	
Center of a rectangle		Centroid of a region	
Length of a line segment		Length of an arc	
Surface area of a cylinder		Surface area of a solid of revolution	
Mass of a solid of constant density		Mass of a solid of variable density	
Volume of a rectangular solid		Volume of a region under a surface	
Sum of a finite number of terms	$a_1 + a_2 + \cdots + a_n = S$	Sum of an infinite number of terms	$a_1 + a_2 + a_3 + \cdots = S$

The ___Area___ problem combines ___differential___ and ___integral___ calculus.

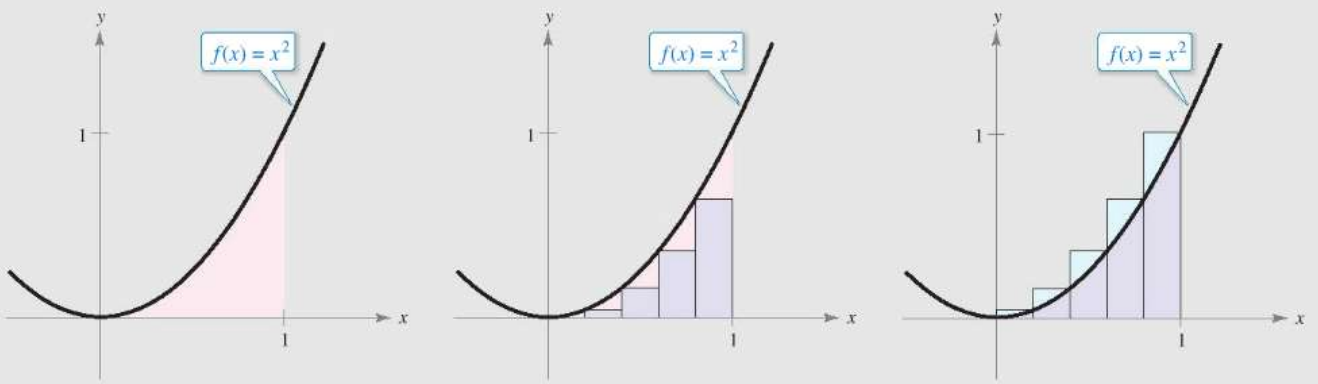
Example 2: Consider the region bounded by the graphs of $f(x)=x^2$, $y=0$, and $x=1$.

Exploration

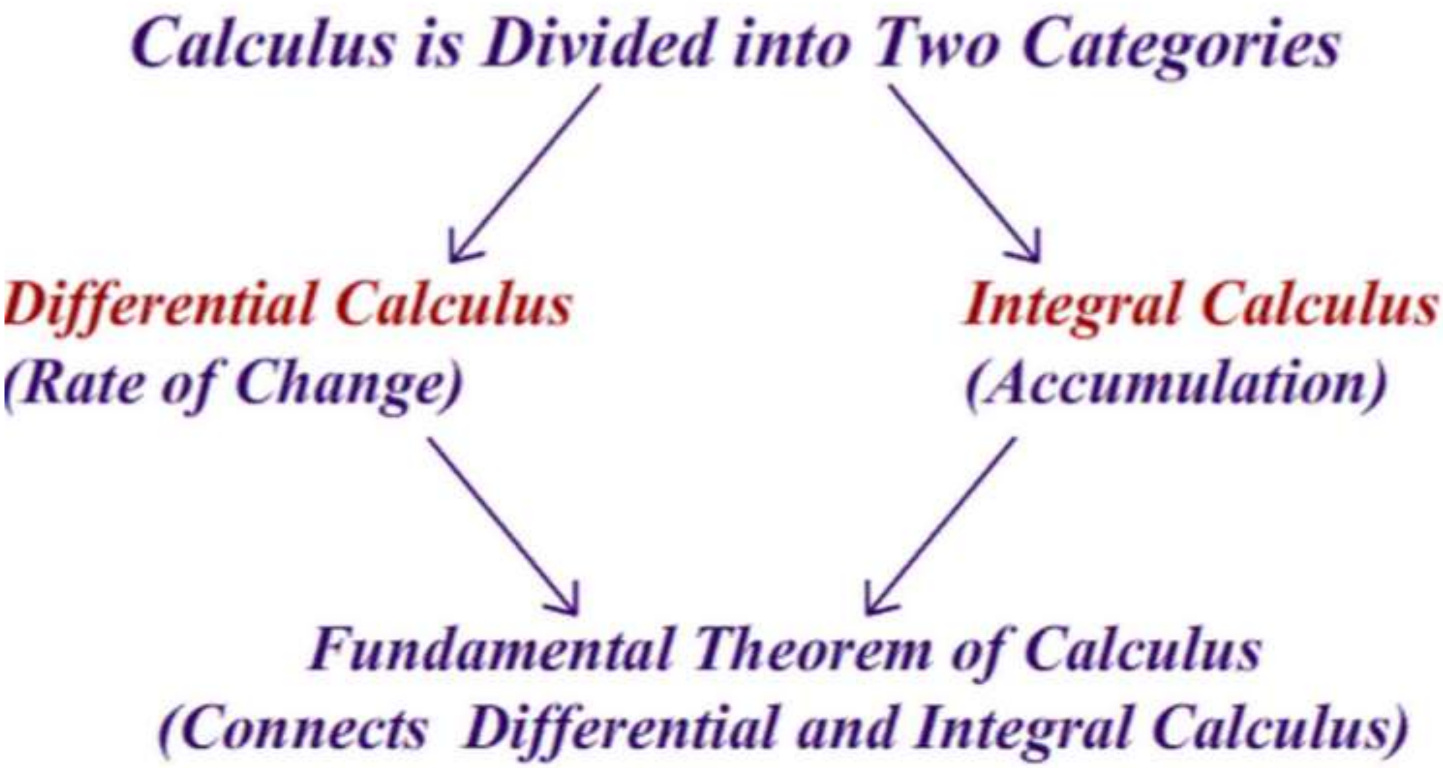
Consider the region bounded by the graphs of

$$f(x) = x^2, \quad y = 0, \quad \text{and} \quad x = 1$$

as shown in part (a) of the figure. The area of the region can be approximated by two sets of rectangles—one set inscribed within the region and the other set circumscribed over the region, as shown in parts (b) and (c). Find the sum of the areas of each set of rectangles. Then use your results to approximate the area of the region.



(a) Bounded region (b) Inscribed rectangles (c) Circumscribed rectangles


Calculus is Divided into Two Categories
Differential Calculus
(Rate of Change)
Integral Calculus
(Accumulation)
Fundamental Theorem of Calculus
(Connects Differential and Integral Calculus)

Section 1.2: Finding Limits Graphically and Numerically

When you finish your homework you should be able to…

- π Estimate a limit using a numerical or graphical approach.
- π Learn different ways that a limit can fail to exist.
- π Study and use a formal definition of limit.

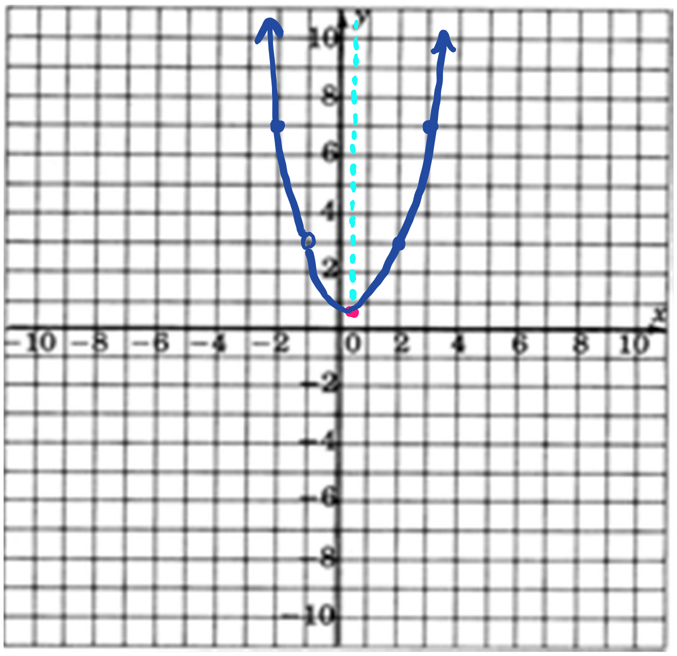
$A^3+B^3=(A+B)(A^2-AB+B^2)$

$A^3-B^3=(A-B)(A^2+AB+B^2)$

Warm-up: Consider the function $f(x)=\dfrac{x^3+1}{x+1}$

$f(x)=\dfrac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}},\ x\neq -1$

1. Graph the function by hand.



vertex

$f(x)=x^2-x+1,\ x\neq -1$

Vertex $=\left(-\frac{b}{2a},\ f\left(-\frac{b}{2a}\right)\right)$

$f(x)=ax^2+bx+c$

$-\frac{b}{2a}=-\frac{(-1)}{2(1)}=\frac{1}{2}=h$

$f\left(\frac{1}{2}\right)=\frac{1}{4}-\frac{1}{2}\cdot\frac{2}{2}+1\cdot\frac{4}{4}=\frac{3}{4}$

$(h,k)=\left(\frac{1}{2},\frac{3}{4}\right)$

axis of symmetry: $x=h$

$x=\frac{1}{2}$

graphing part

Discontinuity

$f(-1)=(-1)^2-(-1)+1=3$ excluded open circle

$f(2)=3$

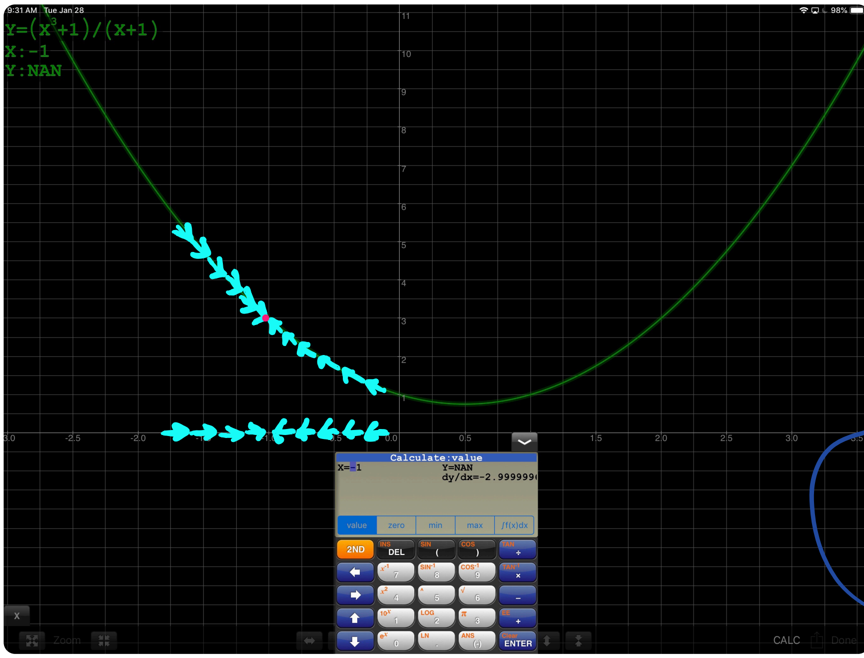
$f(3)=9-3+1=7$

$f(-2)=7$

2. $f(-1) =$ undefined.

At (-1, 3), there's an open circle on graph, due to the discontinuity based on the x+1 divided out from the denominator.

3. Graph the function on your graphing calculator and sketch the result.



Since as x → -1 from the left and right, y is approaching 3,

$$\lim_{x \to -1} \frac{x^3+1}{x+1} = 3$$

4. Now, please complete the chart below.

As $x \to -1$ from the left, $f(x) \to 3$ $\qquad \lim_{x \to -1^-} \frac{x^3+1}{x+1} = 3$

As $x \to -1$ from the right, $f(x) \to 3$ $\qquad \lim_{x \to -1^+} \frac{x^3+1}{x+1} = 3$

x	-1.1	-1.01	-1.001	-0.999	-0.99	-0.9
$f(x) = \dfrac{x^3+1}{x+1}$	3.31	3.0301	3.003001	2.997001	2.9701	2.71

Guess what? We just did some calculus!!! The first three entries on the chart above represent the limit as x approaches -1 from the left of f(x). In mathese, we write this as

$\lim_{x \to -1^-} f(x)$ or for clarity, we write

$\lim_{x \to -1} \frac{x^3+1}{x+1}$. The next three entries on the chart above represent the limit as x approaches -1 from the right of f(x). In mathese, we write this as

$\lim_{x \to -1^+} f(x)$ or for clarity, we write

$\lim_{x \to -1^+} \frac{x^3+1}{x+1}$.

Now…did we get the same approximate result for each one-sided limit? Yes !! :)

Cool! This means that we have discovered that the limit as x approaches -1 of f(x) is 3.

This implies that if there is no designation of approaching ~~form~~ from the left or approaching from the right, the one - sided limits approach the same y - value.

note $(\sqrt{1-x})^2 - (2)^2 = (1-x) - 4$

$(A+B)(A-B) = A^2 - B^2$

Example 2: Complete the table and use the result to estimate the limit.

$\frac{0}{0}$ when $x=-3$

a. $\lim\limits_{x\to -3} \frac{\left(\sqrt{1-x}-2\right)}{x+3} \cdot \frac{\left(\sqrt{1-x}+2\right)}{\sqrt{1-x}+2}$ $\rightarrow$ $= \lim\limits_{x\to -3} \frac{-\cancel{(x+3)}}{\cancel{(x+3)}(\sqrt{1-x}+2)}$

Finding limit analytically $= \lim\limits_{x\to -3} \frac{1-x-4}{(x+3)(\sqrt{1-x}+2)}$

$= \lim\limits_{x\to -3} \frac{-1}{\sqrt{1-x}+2}$

D.S. $= \frac{-1}{\sqrt{1-(-3)}+2} \rightarrow = -\frac{1}{4}$

D.S. = direct substitution

Finding limit using a table

$\lim\limits_{x\to -3^-} \frac{\sqrt{1-x}-2}{x+3} = -0.25$, $\lim\limits_{x\to -3^+} \frac{\sqrt{1-x}-2}{x+3} = -0.25$

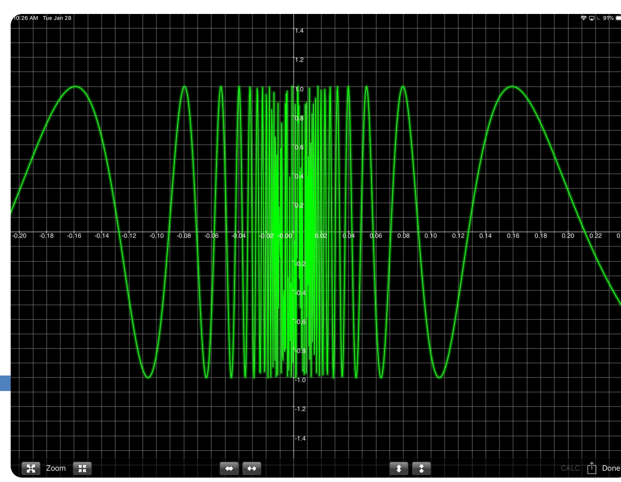
x	-3.1	-3.01	-3.001	-2.999	-2.99	-2.9
$f(x)$	-0.248457	-0.249844	-0.249984	-0.250016	-0.250156	-0.251582

$\boxed{\lim\limits_{x\to -3} \frac{\sqrt{1-x}-2}{x+3} = -0.25}$

b. $\lim\limits_{x\to 0} \cos\frac{1}{x}$ oscillates so $\lim\limits_{x\to 0} \cos\frac{1}{x}$ DNE

x	$1/\pi$	$1/2\pi$	$1/3\pi$	$1/4\pi$	$1/5\pi$	$1/6\pi$
$f(x)$	-1	1	-1	1	-1	1

$\cos\frac{1}{1/\pi} = \cos\pi = -1$



$f(\pm 4) = \frac{1}{7}$

c. Consider the function $f(x) = \dfrac{1}{x^2 - 9}$

Let's graph the function:

$f(2.9) = \frac{1}{8.41-9} = -\frac{1}{.59} = -\frac{1}{\frac{59}{100}}$

VA: $x^2 - 9 = 0$

$x^2 = 9$

$x = \pm 3 \rightarrow x = -3, x = 3$

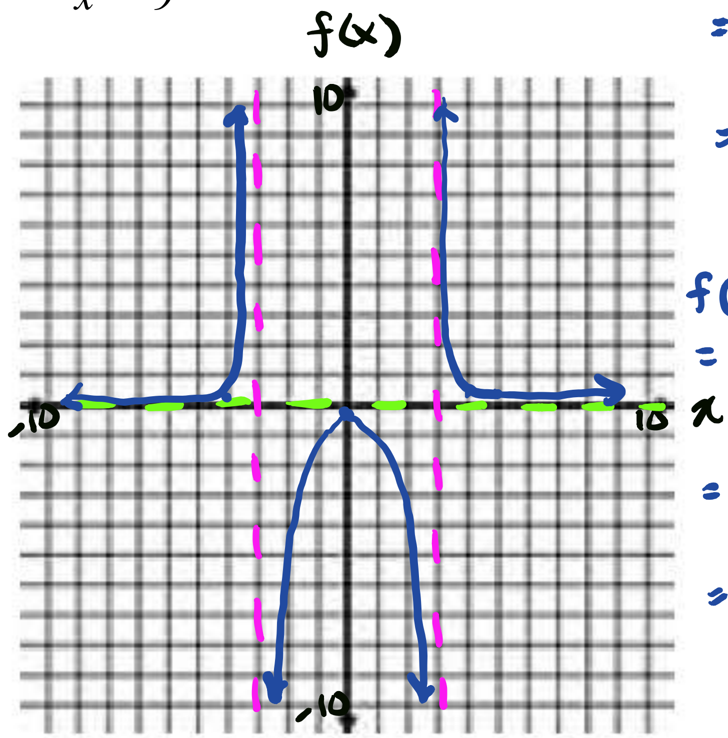
HA: $y = 0$

Intercepts:

y-int: $f(0) = \frac{1}{(0)^2 - 9} = -\frac{1}{9}$

x-int: NONE

$0 = \frac{1}{x^2-9} \rightarrow 0 = 1$ contradiction



$f(3.1) = \frac{1}{9.61-9} = \frac{1}{.61} = \frac{1}{61/100}$

x	2.9	2.99	2.999	3.001	3.01	3.1
$f(x)$	$-100/59$	-16.694	-166.6944	166.638	16.639	$100/61$

What observations would you make about the behavior of this function at the vertical asymptote $x = 3$?

As x approaches 3 from the left of $\frac{1}{x^2-9}$ the function decreases without bound and as x approaches 3 from the right of $\frac{1}{x^2-9}$ the function increases without bound. Therefore, $\lim_{x \to 3^-} \frac{1}{x^2-9} = -\infty$ so DNE and $\lim_{x \to 3^+} \frac{1}{x^2-9} = \infty$ so it do not exist, and thus $\lim_{x \to 3} \frac{1}{x^2-9}$ also does not exist.

COMMON TYPES OF BEHAVIOR ASSOCIATED WITH NONEXISTENCE OF A LIMIT

1. $f(x)$ approaches a different number from the right side of *c* than it approaches from the left side.
2. $f(x)$ increases or decreases without bound as *x* approaches *c*.
3. $f(x)$ oscillates between two fixed values as *x* approaches *c*.

Example 3: Consider the function

$$g(x)=\begin{cases}\sin x, & x\le 0\\ 1-\cos x, & 0<x\le\pi\\ \cos x, & x>\pi\end{cases}$$

a. Please graph the function:

$g(0) = \sin 0 = 0$

$g(-\frac{\pi}{2}) = \sin(-\pi/2) = -1$

Boundary value

$1-\cos 0 = 1-1 = 0$ so open circle at $x=0$

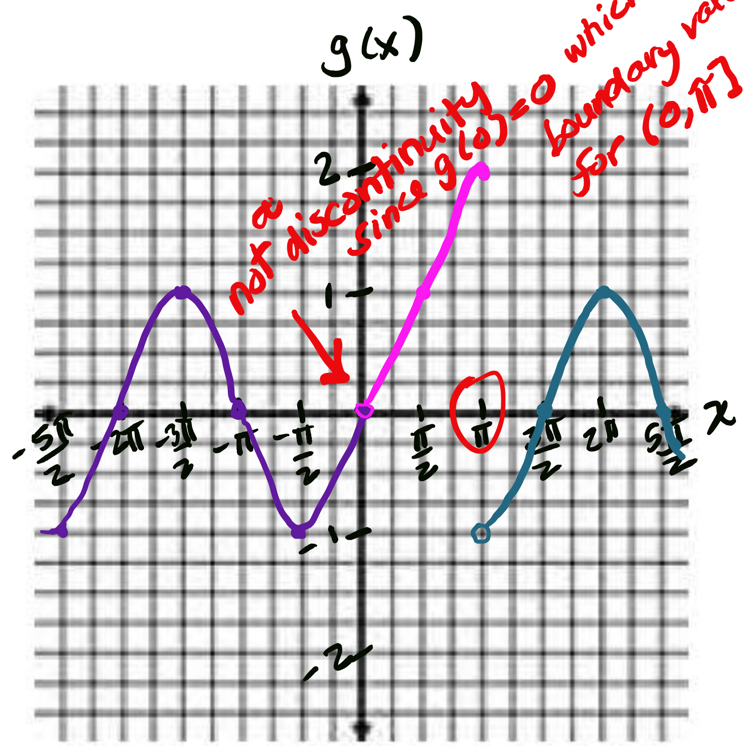
$g(\pi/2) = 1-\cos\pi/2 = 1$

$g(\pi) = 1-\cos\pi = 1-(-1) = 2$

Boundary value

$\cos\pi = -1$ open circle at $x=\pi$

$g(3\pi/2) = \cos 3\pi/2 = 0$



not a discontinuity since $g(0)=0$ which was the boundary value for $(0,\pi]$

b. Now identify the values of *c* for which the $\lim_{x\to c} g(x)$ exists in interval notation.

$\boxed{(-\infty,\pi)\cup(\pi,\infty)}$

$\lim_{x\to\pi^-} g(x) = 2$ and

$\lim_{x\to\pi^+} g(x) = -1$

so $\lim_{x\to\pi} g(x)$ DNE

DEFINITION OF LIMIT

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a **real number**. The statement

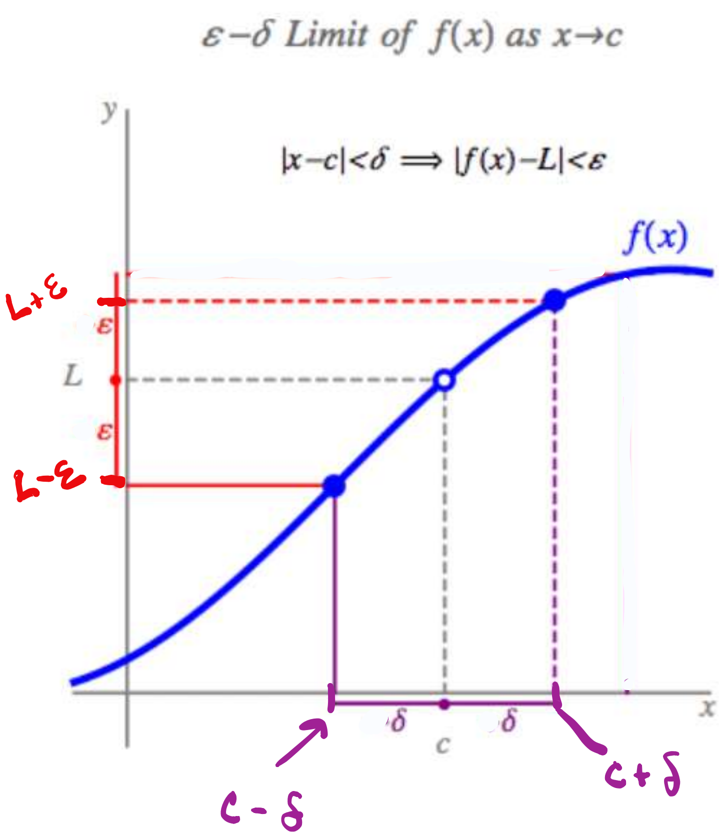
$$\lim_{x \to c} f(x) = L$$

means that for each small positive number epsilon, denoted ε, there exists a small positive number delta, denoted δ, such that if

$$0 < |x - c| < \delta, \text{ then } |f(x) - L| < \varepsilon.$$

Let's check it out!

http://www.acsu.buffalo.edu/~mbschild/Media-Calculus/Media-Calculus.html



Example 4: Find the limit L. Then find $\delta > 0$ such that $|f(x) - L| < 0.01$ whenever $|x - c| < \delta$.

D.S.

$$\lim_{x \to 5}(x^2 + 4) = (5)^2 + 4 = 29$$

$L = 29$

$\varepsilon = 0.01$

$f(x) = x^2 + 4$

$c = 5$

$|ab| = |a||b|$

$|f(x) - L| < 0.01$

$|x^2 + 4 - 29| < 0.01$

$|x^2 - 25| < 0.01$

$|(x+5)(x-5)| < 0.01$

$|x+5||x-5| < 0.01$

$|x-5| < \dfrac{0.01}{|x+5|}$

very close to 5

Let $x = 4.5$:

$\dfrac{0.01}{|4.5+5|} = \dfrac{.01}{9.5} \approx 0.001$

Let $x = 5.5$:

$\dfrac{0.01}{|5.5+5|} = \dfrac{0.01}{10.5} \approx 0.00095$

so $|x-5| < 0.00095$

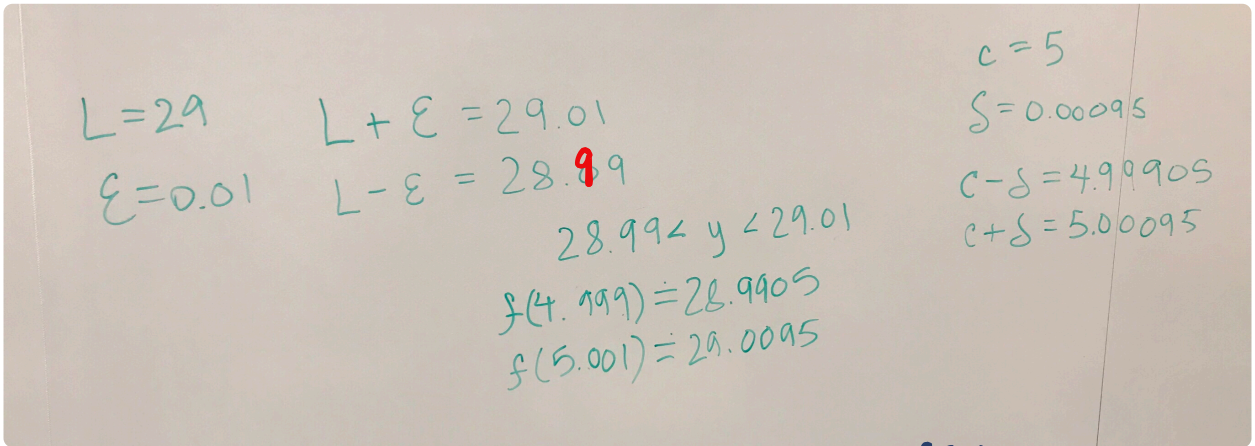
$\delta = 0.00095$

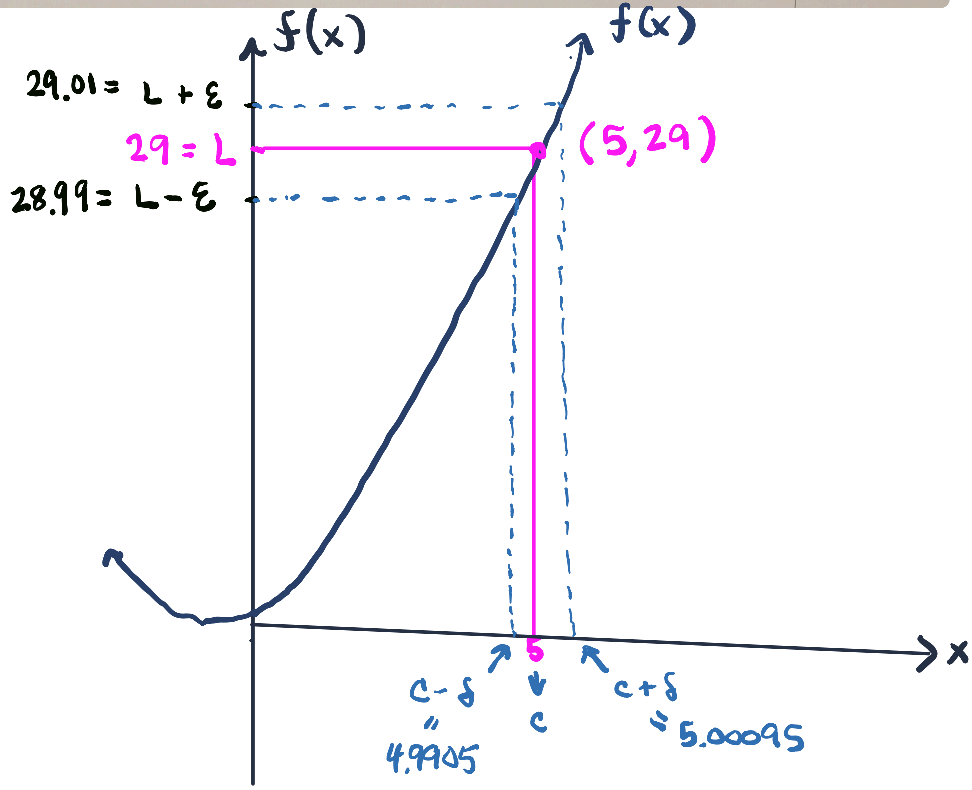
$|ab| = |a||b|$

$\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$

$|x-c| \neq |x| - |c|$

made up math!!


c = 5
δ = 0.00095
L = 29
ε = 0.01
L + ε = 29.01
L − ε = 28.99
c − δ = 4.99905
c + δ = 5.00095
28.99 < y < 29.01
f(4.999) ≐ 28.9905
f(5.001) ≐ 29.0095


f(x)
f(x)
29.01 = L + ε
29 = L
(5,29)
28.99 = L − ε
x
5
c − δ
= 4.9905
c
c + δ
= 5.00095

1.3: Evaluating Limits Analytically

When you are done with your homework you should be able to…

- π Evaluate a limit using properties of limits
- π Develop and use a strategy for finding limits
- π Evaluate a limit using dividing out and rationalizing techniques
- π Evaluate a limit using the Squeeze Theorem

Warm-up:

$8x^3 = (2x)^3,\ 27 = (3)^3$

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

$A^3 + B^3 = (A+B)(A^2 - AB + B^2)$

BTW… $(A+B)^3 = (A+B)(A+B)(A+B)$

1. Factor and simplify.

Factoring

a. $\dfrac{8x^3-27}{2x-3}$

$= \dfrac{\cancel{(2x-3)}[(2x)^2 + (2x)(3) + (3)^2]}{\cancel{2x-3}}$

$= \boxed{4x^2 + 6x + 9}$

$$\begin{array}{r} 4x^2 + 6x + 9 \\ (2x-3)\overline{)\,8x^3 + 0x^2 + 0x - 27} \\ -(8x^3 - 12x^2) \\ \hline +12x^2 + 0x \\ -(12x^2 - 18x) \\ \hline 18x - 27 \\ -(18x - 27) \\ \hline 0 \end{array}$$

So… $(2x-3)(4x^2+6x+9) = 8x^3 - 27$

b. $\underbrace{x^3 - 2x^2}\ \underbrace{-4x + 8}$

$= x^2(x-2) - 4(x-2)$

$= (x-2)(x^2-4)$

$= \boxed{(x-2)(x+2)(x-2)}$

$A^2 - B^2 = (A+B)(A-B)$

c. $3x^2 - 11x - 4$

$= \underbrace{3x^2 - 12x}\ \underbrace{+1x - 4}$

$= 3x(x-4) + 1(x-4)$

$= \boxed{(x-4)(3x+1)}$

$ax^2 + bx + c$

ac

b

−12

−12 +1

−11

a = 3

b = −11

c = −4

$(A+B)(A-B)=A^2-B^2$

2. Rationalize the numerator. Simplify if possible.

$$\frac{(\sqrt{x+1}-2)}{x-3}\cdot\frac{(\sqrt{x+1}+2)}{(\sqrt{x+1}+2)}$$

$$=\frac{(\sqrt{x+1})^2-(2)^2}{(x-3)(\sqrt{x+1}+2)}$$

$$=\frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$$

$$=\frac{\cancel{x-3}^{\,1}}{(\cancel{x-3})_{1}(\sqrt{x+1}+2)}$$

$$=\boxed{\frac{1}{\sqrt{x+1}+2}}$$

Example 1: Find the following function values.

a. $f(x)=-2$

$f(4)=$ $\boxed{-2}$

b. $f(x)=x$

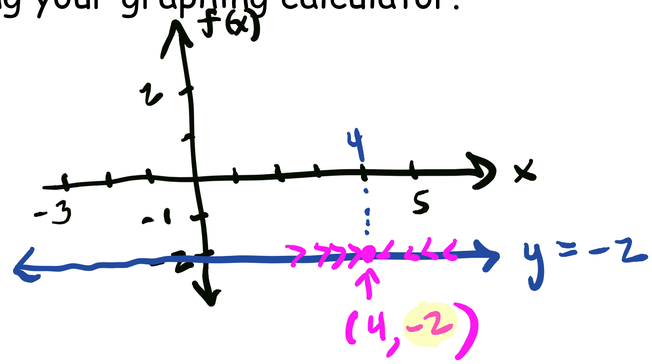
$f(4)=$ $\boxed{4}$

c. $f(x)=x^3$

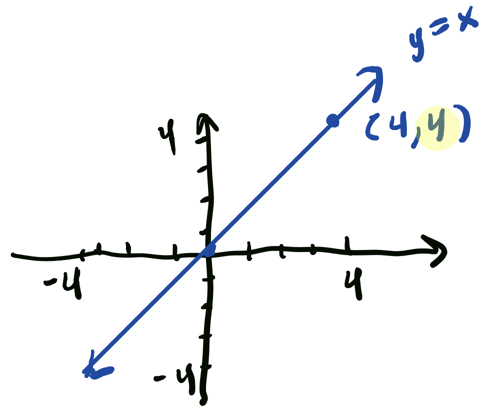
$f(4)=$ $(4)^3=\boxed{64}$

Example 2: Evaluate the following limits using your graphing calculator.

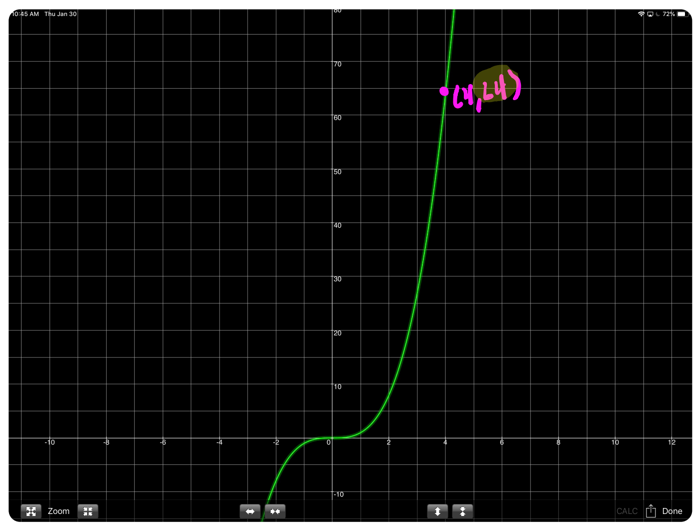
a. $\lim_{x \to 4}(-2) = \boxed{-2}$



b. $\lim_{x \to 4} x = \boxed{4}$



c. $\lim_{x \to 4} x^3 = \boxed{64}$



DIRECT SUBSTITUTION

If the limit of $f(x)$ as x approaches c is $f(c)$, then the limit may be evaluated using **direct substitution**. That is, $\lim_{x \to c} f(x) = f(c)$. These types of functions are continuous at c.

THEOREM: SOME BASIC LIMITS

Let b and c be real numbers and let n be a positive integer.

1. $\lim_{x \to c} b =$ __b__ $y = b$
2. $\lim_{x \to c} x =$ __c__ $y = x$
3. $\lim_{x \to c} x^n =$ __c^n__ $y = x^n$

THEOREM: PROPERTIES OF LIMITS

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$$\lim_{x \to c} f(x) = L \qquad \text{and} \qquad \lim_{x \to c} g(x) = K$$

LIMIT

1. Scalar multiple: $\lim_{x \to c} \left[bf(x) \right] = bL$
2. Sum or difference: $\lim_{x \to c} \left[f(x) \pm g(x) \right] = L \pm K$
3. Product: $\lim_{x \to c} \left[f(x) g(x) \right] = LK$
4. Quotient: $\lim_{x \to c} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{K}, \quad K \neq 0$
5. Power: $\lim_{x \to c} \left[f(x) \right]^n = L^n$

Example 3: Find the limit. Identify the individual functions and the properties you used to evaluate the limit.

a. $\lim_{x\to 2}(5-x)$ $= \lim_{x\to 2} 5 - \lim_{x\to 2} x$

$= 5 - 2$

$= \boxed{3}$

D.S.

$\lim_{x\to 2}(5-x) = 5-2$

$= \boxed{3}$

$f(x) =$ <u>5</u>, $g(x) =$ <u>x</u>

Properties used: #2

b. $\lim_{x\to 0}\left(\frac{6x-5}{x^3-2}\right)$ $= \dfrac{\lim_{x\to 0}(6x-5)}{\lim_{x\to 0}(x^3-2)}$

D.S.

$= \dfrac{6(0)-5}{(0)^3-2}$

$= \boxed{\frac{5}{2}}$

$\lim_{x\to 0}\dfrac{6x-5}{x^3-2}$ D.S. $= \dfrac{6(0)-5}{(0)^3-2}$

$f(x) =$ <u>$6x-5$</u>, $g(x) =$ <u>x^3-2</u>

Properties used: #4

THEOREM: LIMITS OF POLYNOMIAL AND RATIONAL FUNCTIONS

If p is a polynomial and c is a real number, then

$$\lim_{x \to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x \to c} r(x) = r(c).$$

Example 4: Find the following limits.

a. $\lim_{x \to 3} \left(-x^4 + 6x^2 - 2\right)$ D.S. $= -(3)^4 + 6(3)^2 - 2$

$= -81 + 54 - 2$

$= \boxed{-29}$

b. $\lim_{x \to -4} \dfrac{x^3 - 1}{2x + 7}$ D.S. $= \dfrac{(-4)^3 - 1}{2(-4) + 7}$

$= \dfrac{-65}{-1}$

$= \boxed{65}$

THEOREM: THE LIMIT OF A RADICAL FUNCTION

Let n be a positive integer. The following limit is valid for all c if n is odd, and is valid for $c > 0$ if n is even.

$$\lim_{x \to c} \sqrt[n]{x} = \sqrt[n]{c}$$

Example 5: Find the following limits.

a. $\lim_{x \to 225} \sqrt{x}$ DS $= \sqrt{225}$

$= \boxed{15}$

b. $\lim_{x \to -243} \sqrt[5]{x} = \sqrt[5]{-243}$

$= \sqrt[5]{(-3)^5}$

$= \boxed{-3}$

$\sqrt[n]{(x)^m} = x^{m/n}$

$\sqrt[5]{(-3)^5} = (-3)^{5/5} = -3^1 = -3$

THEOREM: THE LIMIT OF A COMPOSITE FUNCTION

Let f and g be functions with the following limits.

$$\lim_{x \to c} g(x) = L \qquad \text{and} \qquad \lim_{x \to L} f(x) = f(L)$$

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

$\sin^2 x = (\sin x)^2$

Example 6: Consider the composite function $h(x) = f(g(x)) = \sin^2 x$. So,

$f(x) =$ x^2 and $g(x) =$ $\sin x$

$h(x) = f[g(x)]$
$= (\sin x)^2$

Evaluate the following.

a. $g\left(\frac{\pi}{4}\right) =$ $\sqrt{2}/2$

b. $\lim\limits_{x \to \boxed{\pi}/\boxed{4}} \left(\boxed{\sin x}\right) = \frac{\sqrt{2}}{2} = L$

c. $\lim\limits_{x \to \frac{\pi}{4}} \sin^2 x = \left(\lim\limits_{x \to \boxed{\pi}/\boxed{4}} \boxed{\sin x}\right)^{\boxed{2}}$

$= L^2$

$= \left(\frac{\sqrt{2}}{2}\right)^2$

$= \boxed{\frac{1}{2}}$

THEOREM: LIMITS OF TRIGONOMETRIC FUNCTIONS

Let c be a real number in the domain of the given trigonometric function.

1. $\lim\limits_{x\to c}(\sin x)=\sin c$
2. $\lim\limits_{x\to c}(\cos x)=\cos c$
3. $\lim\limits_{x\to c}(\tan x)=\tan c$
4. $\lim\limits_{x\to c}(\cot x)=\cot c$
5. $\lim\limits_{x\to c}(\sec x)=\sec c$
6. $\lim\limits_{x\to c}(\csc x)=\csc c$

Example 7: Evaluate the following limits.

a. $\lim\limits_{x\to \frac{2\pi}{3}}(\tan x)$

D.S.

$= \tan\left(\frac{2\pi}{3}\right)$

$= \dfrac{\sin \frac{2\pi}{3}}{\cos \frac{2\pi}{3}}$

$= \dfrac{\sqrt{3}/2}{-1/2}$

$= \boxed{-\sqrt{3}}$

b. $\lim\limits_{x\to \pi}(\cos x) = \cos \pi$

$= \boxed{-1}$

c. $\lim\limits_{x\to \pi}\left(\csc \dfrac{x}{6}\right) = \csc\left[\lim\limits_{x\to \pi} \dfrac{x}{6}\right]$

D.S.

$= \csc \frac{\pi}{6}$

$= \dfrac{1}{\sin \frac{\pi}{6}}$

$= \dfrac{1}{1/2}$

$= \boxed{2}$

THEOREM: FUNCTIONS THAT AGREE AT ALL BUT ONE POINT

Let *c* be a real number and let $f(x)=g(x)$ for all $x \neq c$ in an open interval containing *c*. If the limit of $g(x)$ as *x* approaches *c* exists, then the limit of $f(x)$ also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

Example 8: Find the following limits, first using direct substitution. If you get an indeterminate result, try using the theorem above and identify $g(x)$.

a. Direct substitution: $\lim_{x\to -1} \frac{x^3+1}{x+1}$

1

$\lim_{x\to -1} \frac{x^3+1}{x+1} \overset{D.S.}{=} \frac{(-1)^3+1}{(-1)+1} = \frac{0}{0}$ indeterminate MORE WORK

2 Factor & rewrite

$f(x) = \frac{x^3+1}{x+1}$

$g(x) = \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{x+1}}$

$g(x) = x^2-x+1$

3

$\lim_{x\to -1} \frac{x^3+1}{x+1} = \lim_{x\to -1} (x^2-x+1)$

$\overset{D.S.}{=} (-1)^2-(-1)+1 = \boxed{3}$

b. Direct substitution: $\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = \frac{(3)^2-5(3)+6}{(3)-3} = \frac{0}{0}$ MORE WORK

$\lim_{x\to 3} \frac{x^2-5x+6}{x-3} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{x-3}}$

$= \lim_{x\to 3} (x-2)$

$\overset{D.S.}{=} (3)-2$

$= \boxed{1}$

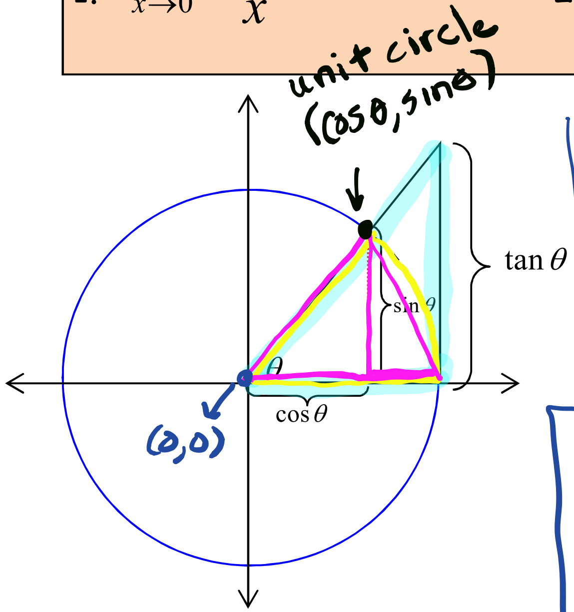
THE SQUEEZE THEOREM

If $h(x) \le f(x) \le g(x)$ for all *x* in an open interval containing *c*, except possibly at *c* itself, and if $\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$ then $\lim_{x\to c} f(x)$ exists and is equal to *L*.

THEOREM: TWO SPECIAL TRIGONOMETRIC LIMITS

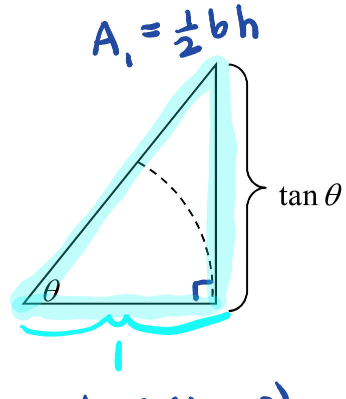
Memorize

1. $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$ 2. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x}=0$



$m=\dfrac{\Delta y}{\Delta x}=\dfrac{\sin\theta-0}{\cos\theta-0}=\tan\theta$

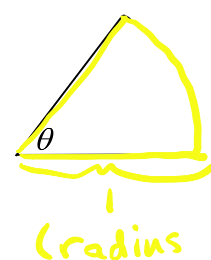
$A_1=\frac{1}{2}bh$



$A_1=\frac{1}{2}(1)(\tan\theta)$

$A_1=\dfrac{\tan\theta}{2}$

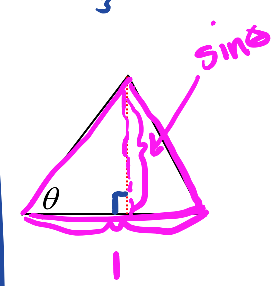
$A_2=\frac{1}{2}\theta r^2$



$A_2=\frac{1}{2}\theta(1)^2$

$A_2=\dfrac{\theta}{2}$

$A_3=\frac{1}{2}bh$



$A_3=\frac{1}{2}(1)(\sin\theta)$

$A_3=\dfrac{\sin\theta}{2}$

$4\geq 3\geq 2 \Leftrightarrow \frac{1}{4}\leq\frac{1}{3}\leq\frac{1}{2}$ Keep in mind

$\dfrac{\frac{1}{1}}{\frac{1}{\cos\theta}}=1\cdot\dfrac{\cos\theta}{1}$
$=\cos\theta$
note

$A_1\geq A_2\geq A_3$

$\dfrac{\tan\theta}{\cancel{2}}\cdot\dfrac{\cancel{2}}{\sin\theta}\geq\dfrac{\theta}{\cancel{2}}\cdot\dfrac{\cancel{2}}{\sin\theta}\geq\dfrac{\cancel{\sin\theta}}{\cancel{2}}\cdot\dfrac{\cancel{2}}{\cancel{\sin\theta}}$

$\dfrac{\cancel{\sin\theta}}{\cos\theta}\cdot\dfrac{1}{\cancel{\sin\theta}}\geq\dfrac{\theta}{\sin\theta}\geq 1$

A_1 A_2 A_3

$\dfrac{1}{\cos\theta}\geq\dfrac{\theta}{\sin\theta}\geq 1$

$\frac{1}{A_1}$ $\cos\theta\leq\dfrac{\sin\theta}{\theta}\leq 1$

$\frac{1}{A_2}$ $\frac{1}{A_3}$

$\lim\limits_{\theta\to 0}\cos\theta\leq\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\leq\lim\limits_{\theta\to 0}1$

D.S $\cos 0\leq\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\leq 1$ D.S

$1\leq\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}\leq 1$

$\therefore$ $\lim\limits_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$ by the squeeze theorem

therefore

Example 9: Find the following limits.

a. Direct Substitution: $\lim_{x\to 0}\frac{\tan^2 x}{x} \overset{D.S.}{=} \frac{(\tan 0)^2}{0} = \frac{0}{0}$ MORE WORK!

$$\lim_{x\to 0}\frac{\tan^2 x}{x} = \lim_{x\to 0}\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{x}$$

$$= \lim_{x\to 0}\left(\frac{\sin x}{\cos^2 x}\cdot\frac{\sin x}{x}\right)$$

$$= \lim_{x\to 0}\frac{\sin x}{\cos^2 x}\cdot\lim_{x\to 0}\frac{\sin x}{x}$$ special trig limit

$$\overset{D.S.}{=} \frac{\sin 0}{(\cos 0)^2}\cdot 1$$

$$= \frac{0}{1}\cdot 1$$

$$= \boxed{0}$$

b. Direct Substitution: $\lim_{x\to \pi/4}\frac{1-\tan x}{\sin x-\cos x} \overset{D.S.}{=} \frac{1-\tan \pi/4}{\sin \pi/4-\cos \pi/4} = \frac{0}{0}$ MORE WORK

$$\lim_{x\to \pi/4}\frac{1-\tan x}{\sin x-\cos x} = \lim_{x\to \pi/4}\frac{1-\frac{\sin x}{\cos x}}{\sin x-\cos x}$$

$$= \lim_{x\to \pi/4}\frac{\cancel{\cos x-\sin x}}{\cos x}\cdot\frac{1}{-1\,\cancel{(\cos x-\sin x)}}$$

$$= \lim_{x\to \pi/4} -\frac{1}{\cos x}$$

$$\overset{D.S.}{=} -\frac{1}{\cos \pi/4} = -\frac{1}{\frac{1}{\sqrt{2}}}$$

$$= \boxed{-\sqrt{2}}$$

STRATEGIES FOR FINDING LIMITS

1. Try using direct substitution first. If this works, you are done! If not, go to step 2.

2. If you obtain an indeterminate result when using direct substitution ($\frac{0}{0}$), try

   a. Factoring the numerator and denominator, dividing out common factors, and then use direct substitution on the new expression.

   b. Rationalize the numerator, and then use direct substitution on the new expression.

3. If you obtain an indeterminate result when using direct substitution (0/0), on a trigonometric expression try

   a. Rewriting the expression using trigonometric identities, and then use direct substitution on the new expression.

   b. Rewriting the expression using trigonometric identities, and then use the special trigonometric limits $\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$.

4. Verify your result by graphing the function on your graphing calculator.

Example 10: Find the following limits.

a. Direct substitution: $\lim_{x \to 4} \frac{\sqrt{x-2}-\sqrt{2}}{x-4}$ D.S. $= \frac{0}{0}$

Evil Plan

Rationalize numerator

$$\lim_{x \to 4} \frac{\left(\sqrt{x-2}-\sqrt{2}\right)}{x-4} \cdot \frac{(\sqrt{x-2}+\sqrt{2})}{(\sqrt{x-2}+\sqrt{2})}$$

$$= \lim_{x \to 4} \frac{(x-2)-2}{(x-4)(\sqrt{x-2}+\sqrt{2})}$$

$$= \lim_{x \to 4} \frac{\cancel{x-4}^{\,1}}{\cancel{(x-4)}_{\,1}(\sqrt{x-2}+\sqrt{2})}$$

$$= \lim_{x \to 4} \frac{1}{\sqrt{x-2}+\sqrt{2}}$$

$$\overset{D.S.}{=} \frac{1}{\sqrt{4-2}+\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}+\sqrt{2}}$$

$$= \boxed{\frac{1}{2\sqrt{2}}} \text{ or } \boxed{\frac{\sqrt{2}}{4}}$$

b. Direct Substitution: $\lim_{\Delta x \to 0} \frac{2(x+\Delta x)^2 - 2x^2}{\Delta x}$ D.S. $= \frac{0}{0}$

Evil Plan

Factor & simplify

or

expand & simplify

$$\lim_{\Delta x \to 0} \frac{2(x+\Delta x)^2 - 2x^2}{\Delta x}$$

$$= 2 \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - (x)^2}{\Delta x}$$

$A^2 - B^2 = (A+B)(A-B)$

$$= 2 \lim_{\Delta x \to 0} \frac{[(x+\Delta x)+x][(x+\Delta x)-x]}{\Delta x}$$

$$= 2 \lim_{\Delta x \to 0} \frac{[2x+\Delta x][\cancel{\Delta x}^{\,1}]}{\cancel{\Delta x}_{\,1}}$$

note $(x+\Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2$

$$= 2 \lim_{\Delta x \to 0} (2x+\Delta x)$$

$$\overset{D.S.}{=} 2(2x+0)$$

$$= \boxed{4x}$$

c. Direct Substitution: $\lim_{x \to 0} \frac{\frac{1}{x-3}+\frac{1}{3}}{x} = \frac{0}{0}$ more work!

Evil Plan

common denom & simplify

$\lim_{x \to 0} \frac{\frac{3}{3}\cdot\frac{1}{(x-3)}+\frac{1}{3}\cdot\frac{(x-3)}{(x-3)}}{x}$

$= \lim_{x \to 0} \frac{3+x-3}{3(x-3)} \cdot \frac{1}{x}$

$= \lim_{x \to 0} \frac{\cancel{x}^{1}}{3(x-3)} \cdot \frac{1}{\cancel{x}_{1}}$

D.S.
$= \frac{1}{3(0-3)}$

$= \boxed{-\frac{1}{9}}$

d. Direct Substitution: $\lim_{\theta \to 0} \frac{\cos\theta \tan\theta}{\theta} \overset{D.S.}{=} \frac{0}{0}$ More work!

Evil Plan

Trig. Identities

$\lim_{\theta \to 0} \frac{\cos\theta \tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\cancel{\cos\theta}^{1}\left(\frac{\sin\theta}{\cancel{\cos\theta}_{1}}\right)}{\theta}$

$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ special trig. limit

$= \boxed{1}$

1.4: Continuity and One-Sided Limits

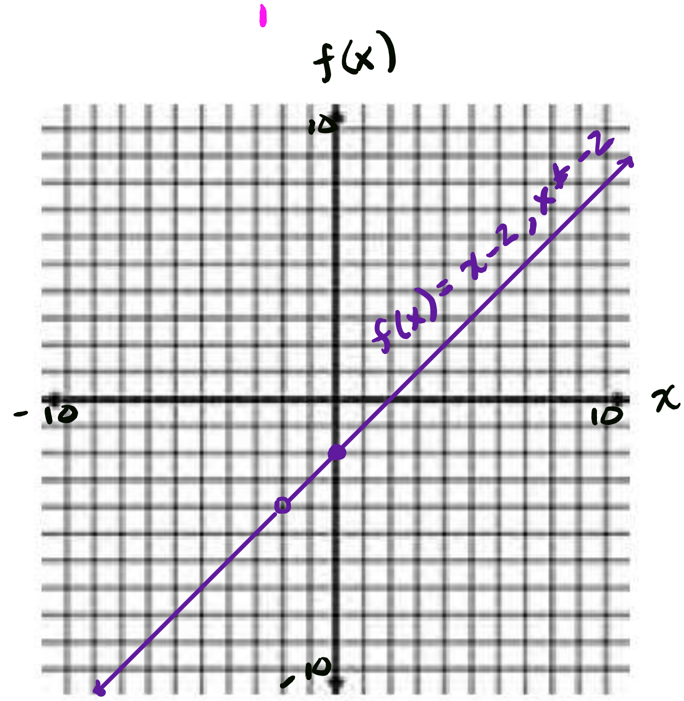
When you are done with your homework you should be able to…

- π Determine continuity at a point and continuity on an open interval
- π Determine one-sided limits and continuity on a closed interval
- π Use properties of continuity
- π Understand and use the Intermediate Value Theorem

Warm-up: Consider the function $f(x) = \dfrac{x^2-4}{x+2}$. $= \dfrac{\cancel{(x+2)}(x-2)}{\cancel{x+2}} = x-2,\ x \neq -2$

a. Sketch the graph.

hole @ $x=-2$: $f(-2) = -4$ DNE



b. $f(-2) =$ undefined

f is not continuous since $f(-2)$ is not defined.

c. $\lim\limits_{x \to -2} \dfrac{x^2-4}{x+2} =$ -4

DEFINITION OF CONTINUITY

CONTINUITY AT A POINT: A function f is **continuous at c** if the following **three** conditions are met.

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

Memorize

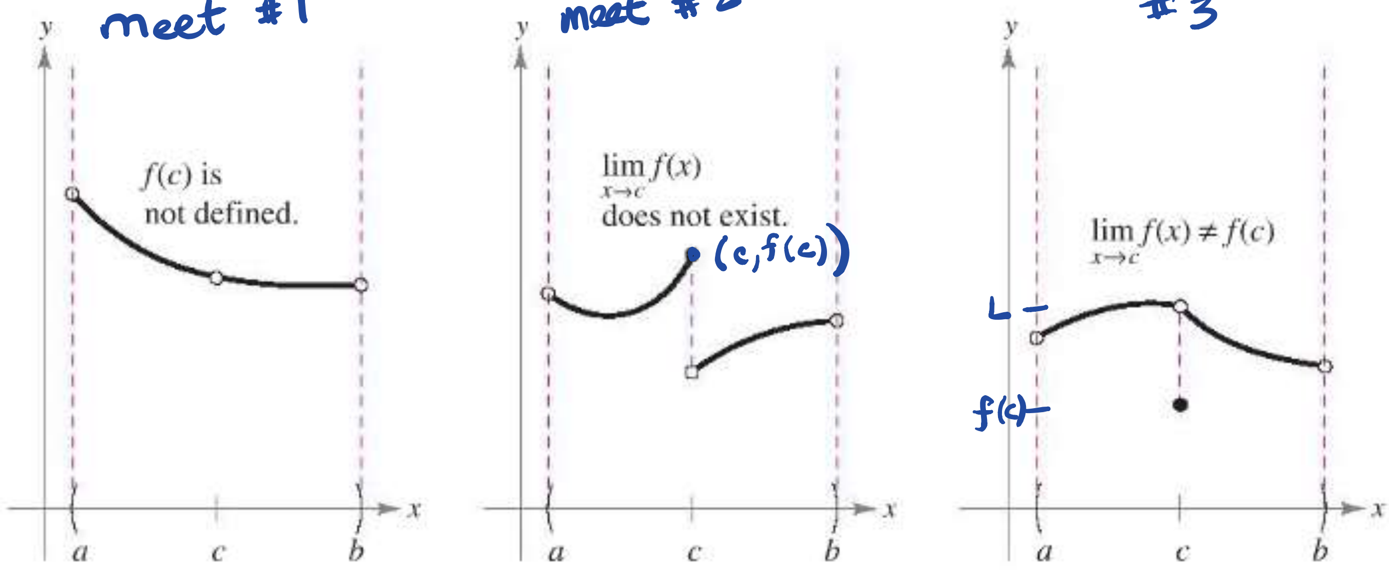
Consider the warm-up problem. Use the **definition of continuity at a point** to determine if f is continuous at $x = -2$?

See above ☺

Doesn't meet #1

Doesn't meet #2

Doesn't meet #3



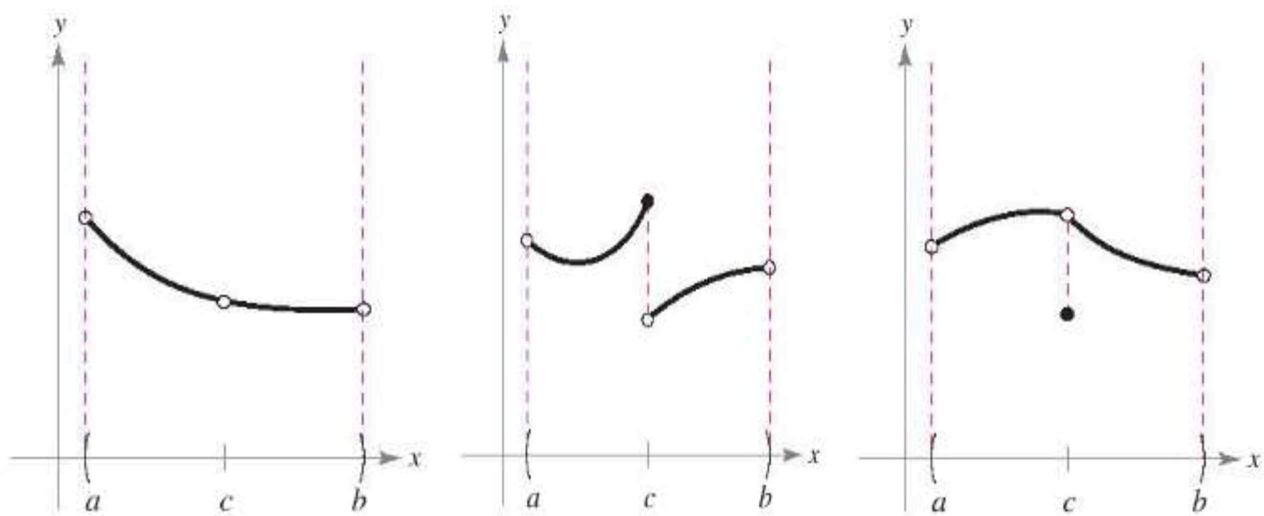
$f(c)$ is undefined

$\lim_{x \to c} f(x)$ DNE

$\lim_{x \to c} f(x) \neq f(c)$

CONTINUITY ON AN OPEN INTERVAL:

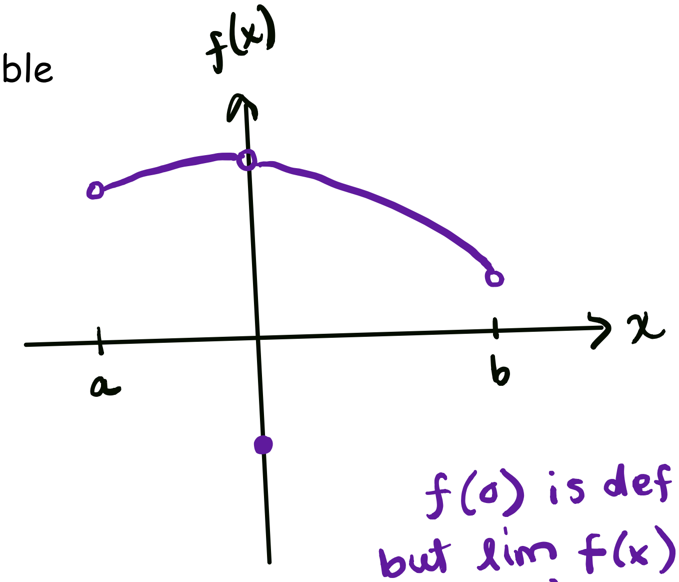
A function is **continuous on an open interval (*a*,*b*)** if it is continuous at **each point in the interval**. A function that is continuous on the entire real line is **everywhere continuous**.



(a) Removable discontinuity **(b)** Nonremovable discontinuity **(c)** Removable discontinuity

Example 1: Draw the graph of the following functions with the given characteristics on the open interval from *a* to *b*:

a. The function has a removable discontinuity at $x = 0$



f(0) is defined but $\lim_{x \to 0} f(x) \neq f(0)$

#3 isn't met

(a, b)

b. The function has a nonremovable discontinuity at $x = 0$

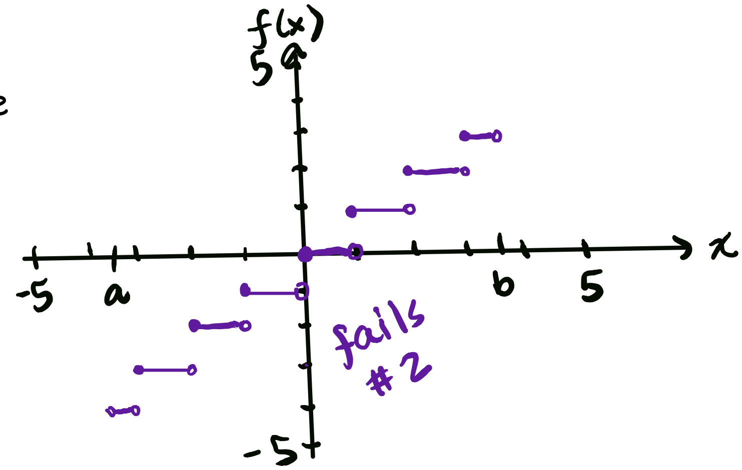
greatest integer function

Step graph

$f(x) = [\![x]\!]$ or $f(x) = \text{int}(x)$

$f(1) = [\![1]\!] = 1$

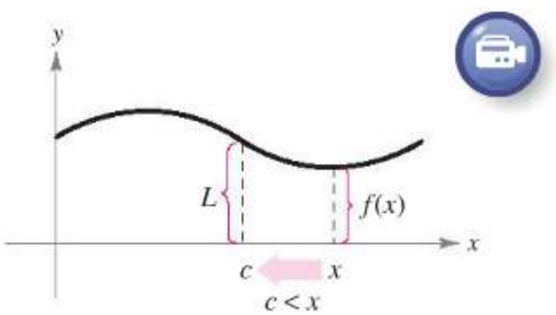
$f(\tfrac{1}{2}) = [\![\tfrac{1}{2}]\!] = 0$



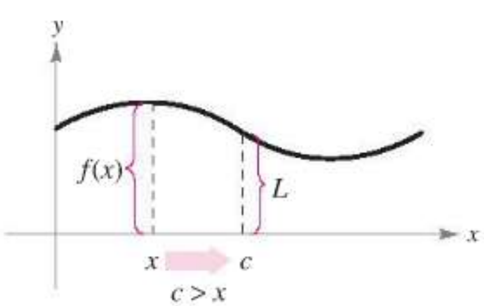
THEOREM: THE EXISTENCE OF A LIMIT

Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$



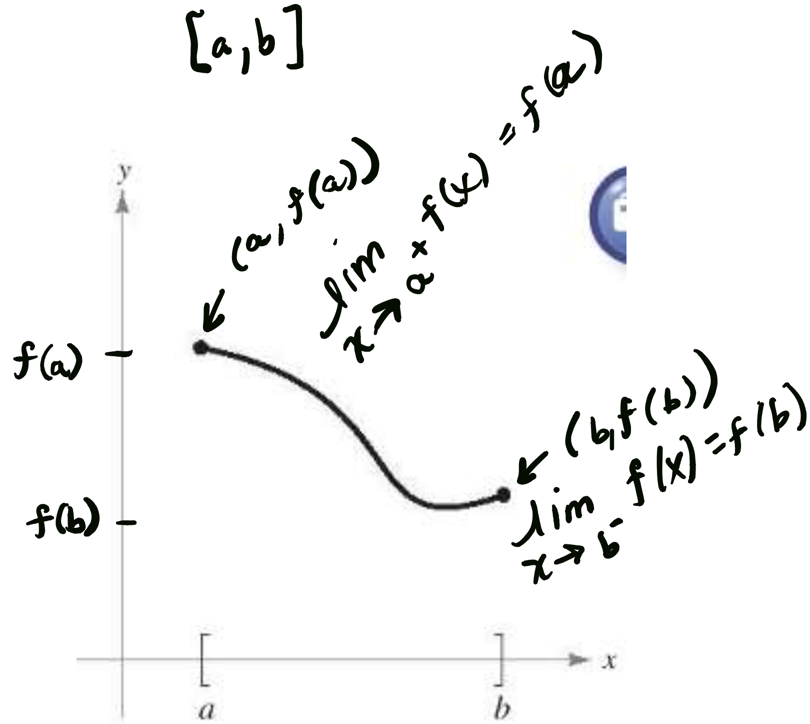
(a) Limit as x approaches c from the right.



(b) Limit as x approaches c from the left.

Figure 1.28

[a, b]



CONTINUITY ON A CLOSED INTERVAL:

A function f is **continuous on the closed interval [a,b]** if it is continuous on the open interval (a, b) and

$$\lim_{x\to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x\to b^-} f(x) = f(b).$$

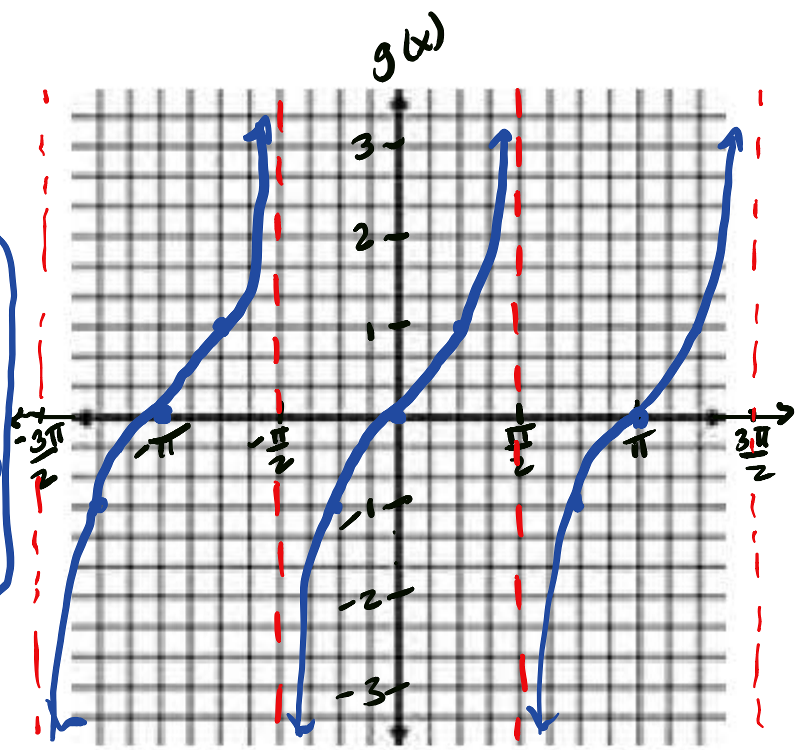
The function f is **continuous from the right at *a*** and **continuous from the left at *b***.

Example 2: Graph each function and use the definition of continuity to discuss the continuity of each function.

a. $g(x) = \tan x = \frac{\sin x}{\cos x}$

$\cos x = 0$ when

$x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

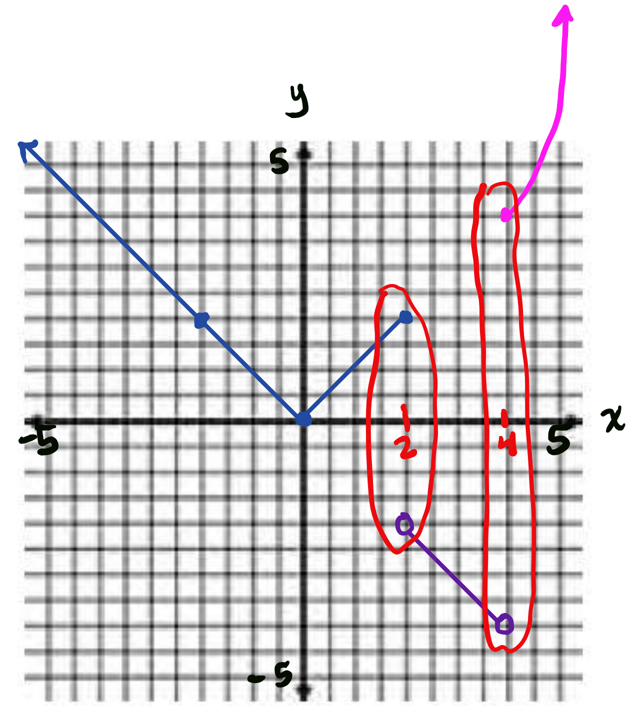


$\boxed{g \text{ is continuous on } \{x : x \in \mathbb{R}, x \neq \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\}}$

$y_1 = |x|$ $\quad y_2 = -x$ $\quad y_3 = \frac{x^2}{4}$

b.

$$y = \begin{cases} |x|, & x \le 2 \\ -x, & 2 < x < 4 \\ \frac{x^2}{4}, & x \ge 4 \end{cases}$$



Discontinuities at $x = 2, 4$

Boundary points

$x = 2$, $y = -2$

$x = 4$, $y = -4$

open circles at $(2, -2)$ and $(4, -4)$

y is continuous at

interval notation → $(-\infty, 2) \cup (2, 4) \cup (4, \infty)$

or

set-builder notation → $\{x : x \in \mathbb{R},\ x \ne 2, 4\}$

THEOREM: PROPERTIES OF CONTINUITY

If b is a real number and f and g are continuous at $x = c$ then the following functions are also continuous at c.

1. Scalar multiple: bf
2. Sum or difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g},\ g(c) \neq 0$

The following types of functions are continuous at every point in their domains.

1. Polynomial functions: $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
2. Rational functions: $r(x) = \frac{p(x)}{q(x)},\ q(x) \neq 0$
3. Radical functions: $f(x) = \sqrt[n]{x}$ $\sqrt{x} = \sqrt[2]{x}$ Rt to find domain $x \geq 0$ for even indices
4. Trigonometric functions: $\sin x,\ \cos x,\ \tan x,\ \cot x,\ \sec x,\ \csc x$

THEOREM: CONTINUITY OF A COMPOSITE FUNCTION

If g is continuous at c and f is continuous at $g(c)$ then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at c.

THEOREM: THE INTERMEDIATE VALUE THEOREM

If f is continuous on the closed interval $[a,b]$ and k is any number between $f(a)$ and $f(b)$, then there is **at least** one number c in (a,b) such that $f(c)=k$.

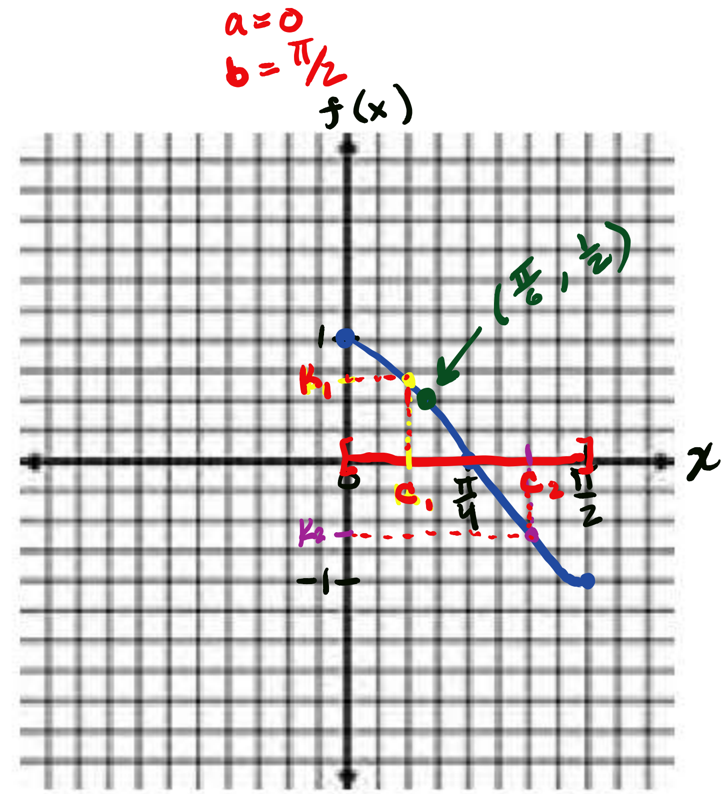
Example 3: Consider the function $f(x)=\cos 2x$ on the closed interval $\left[0,\frac{\pi}{2}\right]$.

a. Sketch the graph of f by hand.

$f(0) = \cos(2\cdot 0) = \cos 0 = 1$

$f(\frac{\pi}{2}) = \cos(2\cdot\frac{\pi}{2}) = \cos\pi = -1$

x	$f(x)$
0	1
$\pi/4$	0
$\pi/2$	-1



b. State the reason why we can apply the intermediate value theorem (IVT).

f is continuous on $[0,\pi/2]$ and $k=\frac{1}{2}$ and $-1<\frac{1}{2}<1$.

c. Use the IVT to find c such that $f(c)=\frac{1}{2}$.

$f(c) = \cos 2c$

$\frac{1}{2} = \cos 2c \rightarrow 2c = \frac{\pi}{3} \rightarrow c = \frac{\pi}{6}$

Example 4: Evaluate the one-sided limits.

a. $\lim\limits_{x\to 2^-}\dfrac{2}{x+2} \overset{D.S.}{=} \dfrac{2}{2+2} = \boxed{\dfrac{1}{2}}$

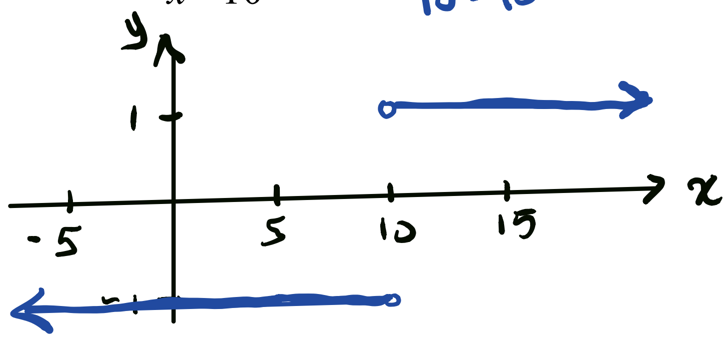
b. $\lim\limits_{x\to 4^-}\dfrac{\sqrt{x}-2}{x-4} \overset{D.S.}{=} \dfrac{\sqrt{4}-2}{4-4} = \dfrac{0}{0}$ MORE WORK!!

$\lim\limits_{x\to 4^-}\dfrac{(\sqrt{x}-2)}{(x-4)}\cdot\dfrac{(\sqrt{x}+2)}{(\sqrt{x}+2)}$

$= \lim\limits_{x\to 4^-}\dfrac{\cancel{x-4}}{\cancel{(x-4)}(\sqrt{x}+2)}$

$= \lim\limits_{x\to 4^-}\dfrac{1}{\sqrt{x}+2} \overset{D.S.}{=} \dfrac{1}{\sqrt{4}+2} = \boxed{\dfrac{1}{4}}$

c. $\lim\limits_{x\to 10^+}\dfrac{|x-10|}{x-10} \overset{D.S.}{=} \dfrac{|10-10|}{10-10} = \dfrac{0}{0}$ More work



$\lim\limits_{x\to 10^+}\dfrac{|x-10|}{x-10} = \boxed{1}$

d. $\lim\limits_{x\to 0^-}\dfrac{1}{x}$ trends to $-\infty$, so the finite limit DNE

let $x = -0.001$

$\dfrac{1}{-\frac{1}{1000}} = -1000$

1.5: Infinite Limits

When you finish your homework you should be able to…

- π Determine infinite limits from the left and from the right
- π Find and sketch the vertical asymptotes of the graph of a function

Warm-up: Evaluate the following limits analytically.

a. $\lim_{x\to 0}\frac{[1/(x+4)-1/4]}{x}$ $\overset{D.S.}{=} \frac{0}{0}$

$\lim_{x\to 0}\frac{\frac{(1)4}{(x+4)4}-\frac{1}{4}\cdot\frac{(x+4)}{(x+4)}}{x}$

$=\lim_{x\to 0}\frac{4-(x+4)}{4(x+4)}\cdot\frac{1}{x}$

$=\lim_{x\to 0}\frac{-\cancel{x}^{1}}{4\cancel{x}_{1}(x+4)}$

$=\lim_{x\to 0}\frac{-1}{4(x+4)}$

$\overset{D.S.}{=}\frac{-1}{4(0+4)} = \boxed{-\frac{1}{16}}$

b. $\lim_{\theta\to 0}\frac{\tan^2\theta}{\theta}$ $=\lim_{\theta\to 0}\frac{\left(\frac{\sin\theta}{\cos\theta}\right)^2}{\theta}$

D.S. 0/0 more work

$=\lim_{\theta\to 0}\frac{\sin^2\theta}{\cos^2\theta}\cdot\frac{1}{\theta}$

$=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos^2\theta}$

$=\left(\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\right)\left(\lim_{\theta\to 0}\frac{\sin\theta}{\cos^2\theta}\right)$

$=(1)\left(\frac{\sin 0}{(\cos 0)^2}\right) = \boxed{0}$

special trig limit

D.S.

c. $\left(\lim_{t\to 0}\frac{\sin(t/2)}{t}\right)\cdot\frac{1/2}{1/2}$

$= \frac{1}{2}\lim_{t\to 0}\frac{\sin(t/2)}{t/2}$

$= \frac{1}{2}\cdot 1$ ← special trig lim

$= \boxed{\frac{1}{2}}$

DEFINITION OF INFINITE LIMITS

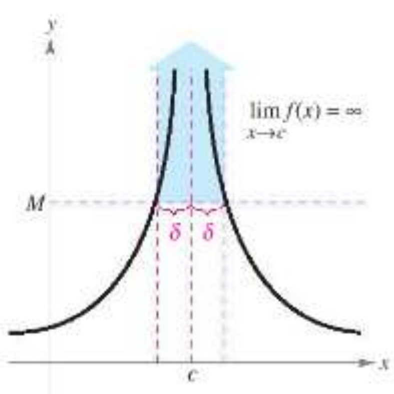
Let f be a function that is defined at every real number in some open interval containing c (except possibly at c). The statement

$$\lim_{x\to c} f(x)=\infty$$

means that for each $M>0$ there exists a $\delta>0$, such that $f(x)>M$ whenever $0<|x-c|<\delta$. Similarly, the statement

$$\lim_{x\to c} f(x)=-\infty$$

means that for each $N<0$ there exists a $\delta>0$, such that $f(x)<N$ whenever $0<|x-c|<\delta$. To define the **infinite limit from the left**, replace $0<|x-c|<\delta$ by $c-\delta<x<c$. To define the **infinite limit from the right**, replace $0<|x-c|<\delta$ by $c<x<c+\delta$.



Example 1: Determine the infinite limit.

a. $\lim_{x \to -3^-} \frac{x}{x^2 - 9}$ D.S. $= \frac{-3}{0}$ $\Rightarrow$ this means there's a vertical asymptote at $x = -3$

Let $x = -3.01$

$\frac{-3.01}{9.0601 - 9} = \frac{-3.01}{\frac{601}{10000}}$

$=$ Big neg. #

$\lim_{x \to -3^-} \frac{x}{x^2 - 9}$ goes to $-\infty$

b. $\lim_{x \to 3^+} \sec \frac{\pi x}{6}$

DEFINITION OF VERTICAL ASYMPTOTE

If $f(x)$ approaches infinity or negative infinity as x approaches c from the right or the left, then the line $x=c$ is a **vertical asymptote** of the graph of f.

THEOREM: VERTICAL ASYMPTOTES

Let f and g be continuous on an open interval containing c. If $f(c)\neq 0$, $g(c)=0$, and there exists an open interval containing c such that $g(x)\neq 0$ for all $x\neq c$ in the interval, then the graph of the function given by

$$h(x)=\frac{f(x)}{g(x)}$$

has a **vertical asymptote** at $x=c$.

Example 2: Find the vertical asymptotes (if any) of the graph of the function.

a. $g(\theta)=\dfrac{\cos\theta}{\theta}$

check out $\lim\limits_{\theta\to 0}\dfrac{\cos\theta}{\theta}\overset{D.S.}{=}\dfrac{1}{0}\Rightarrow$ $\boxed{\text{there's a V.A. at } \theta=0}$

b. $$h(x)=\frac{x^2-4}{x^3+2x^2+x+2}$$

$$h(x)=\frac{(x+2)(x-2)}{x^2(x+2)+1(x+2)}$$

$$h(x)=\frac{(\cancel{x+2})(x-2)}{(\cancel{x+2})(x^2+1)}, \quad x\neq -2$$

$$h(x)=\frac{x-2}{x^2+1}, \quad x\neq -2$$

$x^2+1=0$

$x^2=-1$

imaginary

$\boxed{\text{No vertical asymptotes, just a hole at } x=-2.}$

THEOREM: PROPERTIES OF INFINITE LIMITS

Let *b* and *L* be real numbers and let f and g be functions such that

$$\lim_{x\to c} f(x)=\infty \qquad \text{and} \qquad \lim_{x\to c} g(x)=L$$

1. Sum or difference: $\lim_{x\to c}\left[f(x)\pm g(x)\right]=\infty$
2. Product: $\lim_{x\to c}\left[f(x)g(x)\right]=\infty, \quad L>0$
3. Quotient: $\lim_{x\to c}\left[\frac{g(x)}{f(x)}\right]=0$

Similar properties hold for one-sided limits and for functions for which the limit of $f(x)$ as x approaches c is $-\infty$.

Example 3: Let $\lim_{x \to c} f(x) = -\frac{2}{3}$, $\lim_{x \to c} g(x) = \infty$ and $\lim_{x \to c} h(x) = 5$. Determine the following limits:

a. $\lim_{x \to c} \frac{f(x)}{g(x)}$ $= \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$

$= \frac{-2/3}{\infty}$

$= \boxed{0}$

b. $\lim_{x \to c} \left[f(x) - g(x) \right]^2$

$= \left[\lim_{x \to c} [f(x) - g(x)] \right]^2$

$= \left(\lim_{x \to c} f(x) - \lim_{x \to c} g(x) \right)^2$

$= \left(-\frac{2}{3} - \infty \right)^2$

$= (-\infty)^2$

$= \boxed{\infty}$

c. $\lim_{x \to c} \frac{\left[h(x) f(x) \right]}{3}$

$= \frac{1}{3} \left[\left(\lim_{x \to c} h(x) \right) \left(\lim_{x \to c} f(x) \right) \right]$

$= \frac{1}{3}(5)\left(-\frac{2}{3}\right)$

$= \boxed{-\frac{10}{9}}$

2.1: The Derivative and the Tangent Line Problem

When you are done with your homework you should be able to…

- π Find the slope of the tangent line to a curve at a point
- π Use the limit definition to find the derivative of a function
- π Understand the relationship between differentiability and continuity

Warm-up: Find the following limits.

a. $\lim_{x\to 0}\frac{3x}{x^2+2x}$ $= \lim_{x\to 0}\frac{3\cancel{x}}{\cancel{x}(x+2)}$

$= \lim_{x\to 0}\frac{3}{x+2}$

D.S.

$= \frac{3}{0+2}$

$= \boxed{\frac{3}{2}}$

b. $\lim_{x\to 0}\frac{\frac{4}{4}\frac{1}{(x+4)}-\frac{1}{4}\frac{(x+4)}{(x+4)}}{x} =$

$= \lim_{x\to 0}\frac{4-x-4}{4(x+4)}\cdot\frac{1}{x}$

$= \lim_{x\to 0}\frac{-\cancel{x}}{4(x+4)}\cdot\frac{1}{\cancel{x}}$

$= \lim_{x\to 0}\frac{-1}{4(x+4)}$

D.S.

$= -\frac{1}{4(0+4)}$

$= \boxed{-\frac{1}{16}}$

c. $\lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}$ D.S. $= \frac{0}{0}$ more work!

$A = x + \Delta x$

$B = x$

Diff. of cubes

$A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

$= \lim_{\Delta x \to 0} \frac{(x+\Delta x)(x+\Delta x)(x+\Delta x) - x^3}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + \Delta x^2)(x+\Delta x) - x^3}{\Delta x}$

zero out

$= \lim_{\Delta x \to 0} \frac{\cancel{x^3} + x^2\Delta x + 2x^2\Delta x + 2x\Delta x^2 + \Delta x^2 x + \Delta x^3 \cancel{- x^3}}{\Delta x}$

Pascal's Δ

$(a+b)^0 = 1$

$(a+b)^1 = 1a + 1b$

$(a+b)^2 = 1a^2 + 2ab + 1b^2$

$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}^{1}(3x^2 + 3x\Delta x + \Delta x^2)}{\cancel{\Delta x}_{1}}$

$= \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + \Delta x^2)$

D.S. $= 3x^2 + 3x(0) + (0)^2$

$= \boxed{3x^2}$

DEFINITION OF TANGENT LINE WITH SLOPE *m*

If f is defined on an open interval containing *c*, and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

exists, then the line passing through $(c, f(c))$ with slope m is the **tangent line** to the graph of f at the point $(c, f(c))$.

The slope of the tangent line to the graph of f at the point $(c, f(c))$ is also called the **slope of the graph of *f* at *x* = *c*.

Example 1: Find the slope of the graph of $f(x) = 1 + x^2$ at the point $(1, 2)$ using the limit definition.

c $f(c)$

$$m = \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - 2}{\Delta x}$$

$$m = \lim_{\Delta x \to 0} \frac{1 + (1+\Delta x)^2 - 2}{\Delta x}$$

zero out

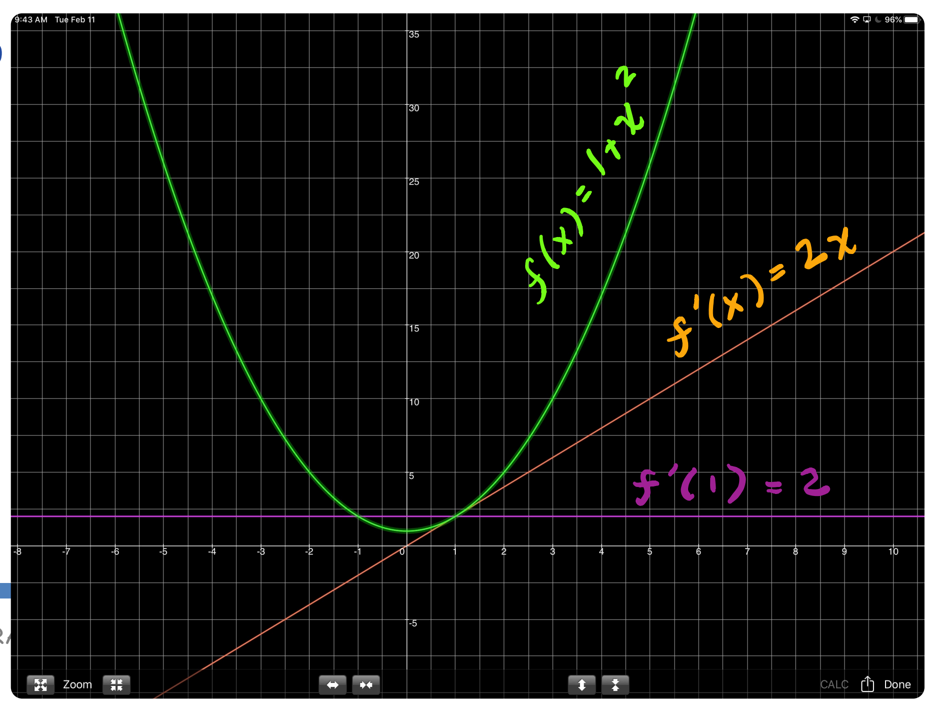
$$m = \lim_{\Delta x \to 0} \frac{\cancel{-1} + \cancel{1} + 2\Delta x + \Delta x^2}{\Delta x}$$

$$m = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}^{1}(2 + \Delta x)}{\cancel{\Delta x}_{1}}$$

$$m = \lim_{\Delta x \to 0} (2 + \Delta x)$$

$$m \overset{D.S.}{=} 2 + 0$$

$$m = 2$$



DEFINITION FOR VERTICAL TANGENT LINES

If f is continuous at *c* and

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} = \infty \quad \text{or} \quad \lim_{\Delta x \to 0} \frac{f(c+\Delta x)-f(c)}{\Delta x} = -\infty$$

the vertical line $x = c$ passing through the point $(c, f(c))$ is a **vertical tangent line** to the graph of f.

DEFINITION OF THE DERIVATIVE OF A FUNCTION

The **derivative** of f at *x* is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

provided the limit exists. For all *x* for which this limit exists, f' is a function of *x*.

The process of finding the derivative of a function is called

differentiation. A function is differentiable at *x* if its

derivative exists at *x* and is differentiable on an open interval (*a*, *b*) if it is

differentiable at every point in the interval.

NOTATION FOR THE DERIVATIVE OF $y = f(x)$:

y', $f'(x)$, $\frac{d}{dx}y \to \frac{dy}{dx}$, $\frac{d}{dx}[f(x)]$, $D_x[f(x)]$

Example 2: Find the derivative of $f(x)=4-x^3$ using the limit process.

$$\frac{d}{dx} f(x) = \frac{d}{dx}(4-x^3)$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{4-(x+\Delta x)^3 - (4-x^3)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{4}-(1\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + 1\Delta x^3) - \cancel{4} + \cancel{x^3}}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-3x^2\Delta x - 3x\Delta x^2 - \Delta x^3}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(-3x^2 - 3x\Delta x - \Delta x^2)}{\cancel{\Delta x}}$$

$$f'(x) = \lim_{\Delta x \to 0} (-3x^2 - 3x\Delta x - \Delta x^2)$$

$$f'(x) \overset{D.S.}{=} -3x^2 - 3x(0) - (0)^2$$

$$\boxed{f'(x) = -3x^2}$$

This is the slope of the graph at $x = x$

What is the slope of the graph at:

a) $x=5$

$f'(5) = -3(5)^2$

$= \boxed{-75}$

b) $x=0$

$f'(0) = -3(0)^2 = \boxed{0}$

ALTERNATIVE LIMIT FORM OF THE DERIVATIVE

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

This form of the derivative requires that the one-sided limits

$\lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$ and $\lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ exist and are equal.

Example 3: Is the function $f(x) = x^{2/3}$ differentiable at $x = 0$?

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

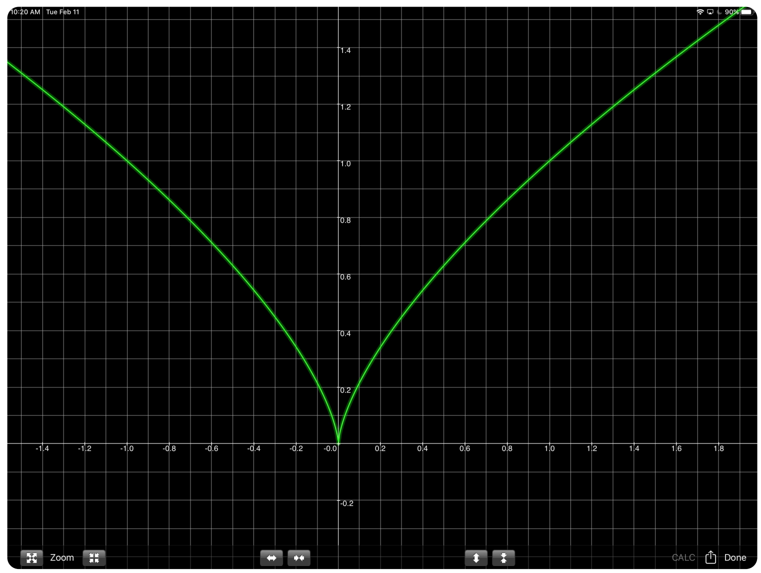
$$= \lim_{x \to 0^+} \frac{x^{2/3} - 0^{2/3}}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{1}{x^{1/3}}$$

tends towards ∞

so $\lim_{x \to 0^-} \frac{1}{x^{1/3}}$ tends towards $-\infty$

thus $\lim_{x \to 0} \frac{1}{x^{1/3}}$ DNE and f is not differentiable at $x = 0$.



note:

$\frac{x^m}{x^n} = x^{m-n}$

$\frac{x^{2/3}}{x^1} = x^{2/3 - 3/3} = x^{-1/3}$

$x^{-a} = \frac{1}{x^a}$

THEOREM: DIFFERENTIABILITY IMPLIES CONTINUITY

If f is differentiable at $x = c$, then f is continuous at $x = c$.

2.2: Basic Differentiation Rules and Rates of Change

When you are done with your homework you should be able to…

- π Find the derivative of a function using the constant rule
- π Find the derivative of a function using the power rule
- π Find the derivative of a function using the constant multiple rule
- π Find the derivative of a function using the sum and difference rules
- π Find the derivative of the sine function and of the cosine function
- π Use derivatives to find rates of change

Warm-up: Find the following derivatives using the limit definition of the derivative.

a. $f(x)=2$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-2}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{0}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0} 0$$

$$\boxed{f'(x)=0}$$

b. $f(x)=x^2$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^2-x^2}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{\cancel{x^2}+2x\Delta x+\Delta x^2-\cancel{x^2}}{\Delta x}$$

zero out

$$f'(x)=\lim_{\Delta x\to 0}\frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}$$

$$f'(x)\overset{D.S.}{=}2x+0$$

$$\boxed{f'(x)=2x}$$

$$\boxed{2x=2x^1 \text{ note}}$$

Manny +1

Evil Plan

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta} = 0$

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

c. $f(x) = \cos x$

$$f'(x) = \lim_{\Delta x\to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{\cos(x+\Delta x) - \cos x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{-\cos x + \cos x\cos\Delta x - \sin x\sin\Delta x}{\Delta x}$$

$$f'(x) = \lim_{\Delta x\to 0} \frac{-\cos x(1-\cos\Delta x)}{\Delta x} - \lim_{\Delta x\to 0} \frac{\sin x\sin\Delta x}{\Delta x}$$

$$f'(x) = \left[\lim_{\Delta x\to 0}(-\cos x)\right]\left[\lim_{\Delta x\to 0}\frac{1-\cos\Delta x}{\Delta x}\right] - \left[\lim_{\Delta x\to 0}\sin x\right]\left[\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}\right]$$

$$f'(x) = [-\cos x][0] - [\sin x][1] \rightarrow \text{so } \boxed{f'(x) = -\sin x}$$

D.S. special trig lim. D.S. special trig lim

$\frac{(2)(3)}{5} = 2\left(\frac{3}{5}\right)$ note

THEOREM: THE CONSTANT RULE

The derivative of a constant function is zero. That is, if c is a real number, then

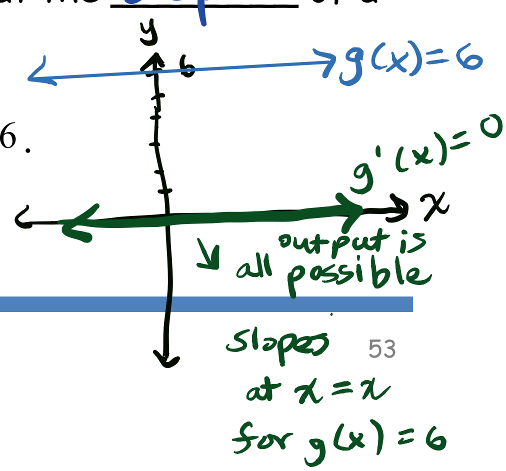
$$\frac{d}{dx}[c] = 0$$

Hmmm…isn't this theorem the equivalent of saying that the slope of a horizontal line is zero?

Example 1: Find the derivative of the function $g(x) = 6$.

$$\frac{d}{dx}[g(x)] = \frac{d}{dx}[6]$$

$$\boxed{g'(x) = 0}$$



THEOREM: THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing zero.

differentiate

Example 2: Find the following derivatives.

with respect to x

a. $\frac{d}{dx} f(x) = \frac{d}{dx}(x^5)$

$f'(x) = 5(x)^{5-1}$

$$\boxed{f'(x) = 5x^4}$$

b. $\frac{d}{dx} f(x) = \frac{d}{dx} x^{1/2}$

$f'(x) = \frac{1}{2}(x)^{\frac{1}{2}-1}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$$\boxed{f'(x) = \frac{1}{2x^{1/2}}}$$

c. $\frac{d}{dx} f(x) = \frac{d}{dx} x^{-5/3}$

$f'(x) = -\frac{5}{3}(x)^{-\frac{5}{3}-1}$

$f'(x) = -\frac{5}{3}x^{-8/3}$

$$\boxed{f'(x) = -\frac{5}{3x^{8/3}}}$$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is a differentiable function and c is a real number, then cf is also differentiable and

$$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$$

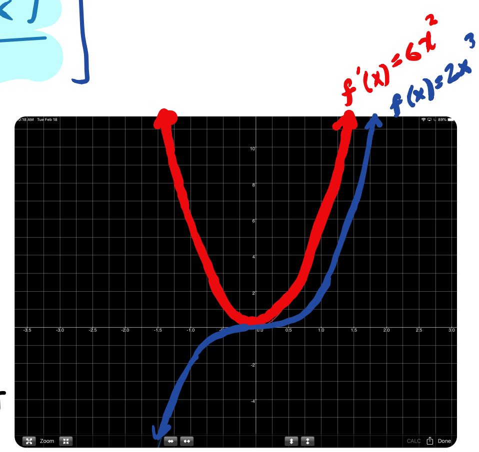
Proof:

$$\frac{d}{dx}\left[cf(x)\right] = \lim_{\Delta x \to 0} \frac{cf(x+\Delta x) - cf(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{c\left[f(x+\Delta x) - f(x)\right]}{\Delta x}$$

$$= \left[\lim_{\Delta x \to 0} c\right]\left[\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right]$$

$$= cf'(x) \quad //$$



Example 3: Find the slope of the graph of $f(x) = 2x^3$ at

$$\frac{d}{dx} f(x) = \frac{d}{dx} 2x^3$$

$$f'(x) = 2\left[\frac{d}{dx} x^3\right] \quad \text{constant mult. rule}$$

$$f'(x) = 2(3x^2) \quad \text{power rule}$$

function of slopes at x=x → $f'(x) = 6x^2$

a. $x = 2$

$$f'(2) = 6(2)^2$$

$$\boxed{f'(2) = 24}$$

b. $x = -6$

$$f'(-6) = 6(-6)^2$$

$$\boxed{f'(-6) = 216}$$

c. $x = 0$

$$f'(0) = 6(0)^2$$

$$\boxed{f'(0) = 0}$$

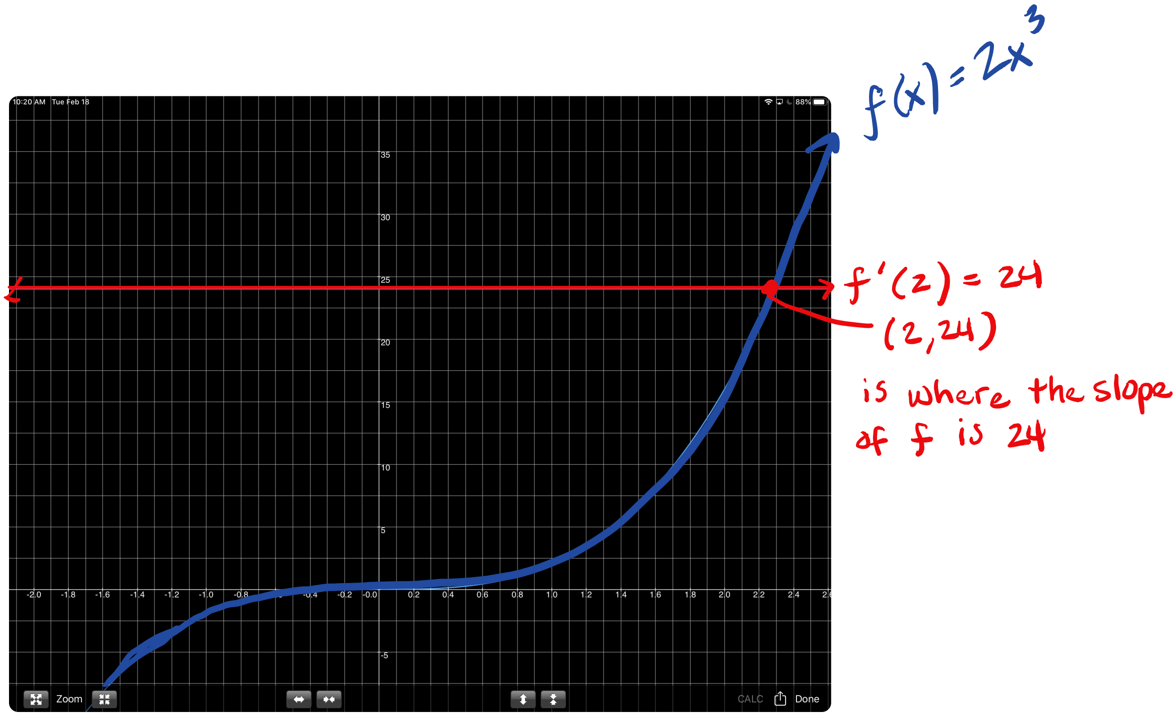
Find the equation of the line tangent to $f(x) = 2x^3$ at $x = 2$.

$$y - y_1 = m(x - x_1)$$

$$y - y_1 = 24(x-2)$$

$$f(2) = 2(2)^3 = 16$$

$$\boxed{y - 16 = 24(x-2)}$$


$f(x) = 2x^3$
$f'(2) = 24$
$(2, 24)$
is where the slope of f is 24

THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$$

Proof (for $f+g$):

$$\frac{d}{dx}\left[f(x)+g(x)\right]=\lim_{\Delta x\to 0}\frac{\left[f(x+\Delta x)-f(x)\right]+\left[g(x+\Delta x)-g(x)\right]}{\Delta x}$$

$$=\left[\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\right]+\left[\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$

$$=f'(x)+g'(x) \quad //$$

Example 4: Find the equation of the line tangent to the graph of $f(x) = x - \sqrt[2]{x}$ at $x = 4$.

1. Find the derivative at $x=4$.

$\frac{d}{dx} f(x) = \frac{d}{dx}(x - x^{1/2})$

$f'(x) = \frac{d}{dx} x^1 - \frac{d}{dx} x^{1/2}$

$f'(x) = 1x^{1-1} - \frac{1}{2} x^{1/2 - 1}$

$f'(x) = 1 - \frac{1}{2x^{1/2}}$

$f'(4) = 1 - \frac{1}{2\sqrt{4}}$

$f'(4) = 1 - \frac{1}{4} \rightarrow f'(4) = \frac{3}{4}$

2. Find $f(4)$.

$f(4) = 4 - \sqrt{4}$

$f(4) = 2$

3. Write equation of the line tangent to f at $x=4$.

$y - y_1 = m(x - x_1)$

$\boxed{y - 2 = \frac{3}{4}(x - 4)}$

THEOREM: DERIVATIVES OF THE SINE AND COSINE FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$

Example 5: Find the derivative of the following functions with respect to the independent variable:

a. $f(x) = \frac{\sin x}{6}$

$\frac{d}{dx} f(x) = \frac{d}{dx}\left(\frac{1}{6}\sin x\right)$

$f'(x) = \frac{1}{6} \frac{d}{dx} \sin x$

constant multiple rule

$\boxed{f'(x) = \frac{1}{6}\cos x}$

b. $\frac{d}{d\theta} r(\theta) = \frac{d}{d\theta}(5\theta - 3\cos\theta)$

$\frac{d\theta}{d\theta} = 1$

$r'(\theta) = 5\frac{d}{d\theta}\theta - 3\frac{d}{d\theta}\cos\theta$

diff. of 2 functions rule and constant multiple rule

$r'(\theta) = 5 \cdot 1 - 3(-\sin\theta)$

$\boxed{r'(\theta) = 5 + 3\sin\theta}$

RATES OF CHANGE

We have seen how the derivative is used to determine slope. The derivative may also be used to determine the rate of change of one variable with respect to another.

A common use for rate of change is to describe the motion of an object moving in a straight line. In such problems, it is customary to use either a horizontal or a vertical line with a designated origin to represent the line of motion. On such lines, movement to the right or upwards is considered to be in the positive direction, and movement to the left or downwards is considered to be in the negative direction.

THE POSITION FUNCTION is denoted by s and gives the position (relative to the origin) of an object as a function of time. If, over a period of time Δt, the object changes its position by $\Delta s = s(t + \Delta t) - s(t)$, then, by the familiar formula

$$\text{rate} = \frac{\text{distance}}{\text{time}}$$

the **average velocity** is

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\Delta s}{\Delta t}$$

Example 6: A ball is thrown straight down from the top of a 220-foot building with an initial velocity of -22 feet per second. The position function for free-falling objects measured in feet is $s(t) = -16t^2 + v_0 t + s_0$.

$s(t) = -16t^2 + (-22)t + 220 \rightarrow s(t) = -16t^2 - 22t + 220$

What is its velocity after 3 seconds?

$\frac{d}{dt} s(t) = \frac{d}{dt}(-16t^2 - 22t + 220)$

$s'(t) = -32t - 22 + 0$

$v(t) = -32t - 22$

$v(3) = -32(3) - 22$

$v(3) = -118$ ft/s

$\boxed{\text{After 3 seconds, the velocity of the ball is } -118 \text{ ft/s.}}$

What is its velocity after falling 108 feet?

$108 = -16t^2 - 22t + 220$

$0 = -16t^2 - 22t + 112$

calculator:

$t = 2.0$

(discard neg. t values)

finding t after falling 108 ft

(2.0, 0)

$v(t) = -32t - 22$

$v(2.0) = -32(2.0) - 22$

$v(2.0) = -86$ ft/s

finding velocity at the time corresponding to falling 108 feet.

$\boxed{\text{The velocity of the ball after falling 108 ft is } -86 \text{ ft/s.}}$

OUT SICK -- Video for 2.3 was assigned

2.3: The Product and Quotient Rules and Higher-Order Derivatives

When you are done with your homework you should be able to…

- π Find the derivative of a function using the product rule
- π Find the derivative of a function using the quotient rule
- π Find the derivative of a trigonometric function
- π Find a higher-order derivative of a function

Warm-up: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

$a^m a^n = a^{m+n}$

Algebra

a. $f(x) = \dfrac{3x^2 - x + 2}{\sqrt{x}}$

$f(x) = (3x^2 - x + 2)x^{-1/2}$

$\frac{d}{dx} f(x) = \frac{d}{dx}(3x^{3/2} - x^{1/2} + 2x^{-1/2})$

$f'(x) = \frac{9}{2}x^{3/2-1} - \frac{1}{2}x^{1/2-1} - 1x^{-1/2-1}$

$f'(x) = \frac{9}{2}x^{1/2} - \frac{1}{2}x^{-1/2} - x^{-3/2}$

$f'(x) = \frac{x^{3/2}}{x^{3/2}} \cdot \frac{9x^{1/2}}{2} - \frac{x}{x} \cdot \frac{1}{2x^{1/2}} - \frac{1}{x^{3/2}} \cdot \frac{2}{2}$

$$\boxed{f'(x) = \frac{9x^2 - x - 2}{2x^{3/2}}}$$

Algebra

b. $g(x) = (5x-3)^2$

$g(x) = 25x^2 - 30x + 9$

$g'(x) = 50x - 30 + 0$

$$\boxed{g'(x) = 50x - 30}$$

trig. identity

c. $f(x) = \cos\left(x - \frac{\pi}{4}\right)$

THEOREM: THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the derivative of the first function times the second function, plus the first function times the derivative of the second function.

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + f(x)g'(x)$$

This rule extends to cover products of more than two factors. For example the derivative of the product of functions $fghk$ is

$$\frac{d}{dx}[fghk] = f'(x)g(x)h(x)k(x) + f(x)g'(x)h(x)k(x) + f(x)g(x)h'(x)k(x) + f(x)g(x)h(x)k'(x)$$

Proof:

Example 1: Find the derivative of the following functions with respect to the independent variable. Simplify your result to a single rational expression with positive exponents.

a. $g(x) = x\cos x$

b. $h(t) = \left(3 - \sqrt{t}\right)^2$

c. $f(x)=(x^3-x)(x^2+2)(x^2+x-1)$ Do not simplify this guy☺

THEOREM: THE QUOTIENT RULE

The quotient of two differentiable functions f and g is itself differentiable at all values of x for which $g(x)\neq 0$. Moreover, the derivative of f/g is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Example 2: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $g(x) = \dfrac{x^4}{x+1}$

b. $h(t) = \dfrac{t}{\sqrt{t} - 1}$

c. $f(x) = \tan x$

THEOREM: DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

Example 3: Find the derivative of the trigonometric functions with respect to the independent variable.

a. $g(x) = -2\csc x$

b. $h(t) = \cot^2 t$

c. $r(s) = \dfrac{\sec s}{s}$

HIGHER ORDER DERIVATIVES

Recall that you can obtain _______________ by differentiating a position function.
You can obtain an _______________ function by differentiating a velocity function. Or, you could also think about the acceleration function as the ______________ derivative of the _______________ function.

Example 4: An automobile's velocity starting from rest is $v(t)=\dfrac{100t}{2t+15}$ where *v* is measured in feet per second. Find the acceleration at

a. 5 seconds

b. 10 seconds

c. 20 seconds

NOTATION FOR HIGHER-ORDER DERIVATIVES

First derivative					
Second derivative					
Third derivative					
Fourth derivative					
***n*th derivative**					

Example 5: Find the given higher-order derivative.

a. $f(x) = 2 - \frac{2}{x}, \quad f'''(x)$

$$h(x) = f(x)g(x)$$
$$h'(x) = [f'(x)]g(x) + [f(x)]g'(x)$$

b. $f^{(4)}(x) = \tan x, \quad f^{(6)}(x)$

$$f^{(5)}(x) = \sec^2 x$$

product rule

$$f^{(5)}(x) = (\sec x)(\sec x)$$

$$f^{(6)}(x) = \frac{d}{dx}(\sec x)\sec x + \sec x \frac{d}{dx}(\sec x)$$

$$f^{(6)}(x) = \sec x \tan x \sec x + \sec x \sec x \tan x$$

$$\boxed{f^{(6)}(x) = 2\sec^2 x \tan x}$$

Chain Rule

$$f^{(5)}(x) = (\sec x)^2$$

$$f^{(6)}(x) = f'(u)\frac{du}{dx}$$

$$f^{(6)}(x) = 2u \sec x \tan x$$

$$f^{(6)}(x) = 2\sec x \sec x \tan x$$

$$\boxed{f^{(6)}(x) = 2\sec^2 x \tan x}$$

$$\frac{du}{dx} = \frac{d\sec x}{dx} \qquad \frac{d}{du}f(u) = \frac{du^2}{du}$$

$$\frac{du}{dx} = \sec x \tan x \qquad f'(u) = 2u$$

2.4: The Chain Rule

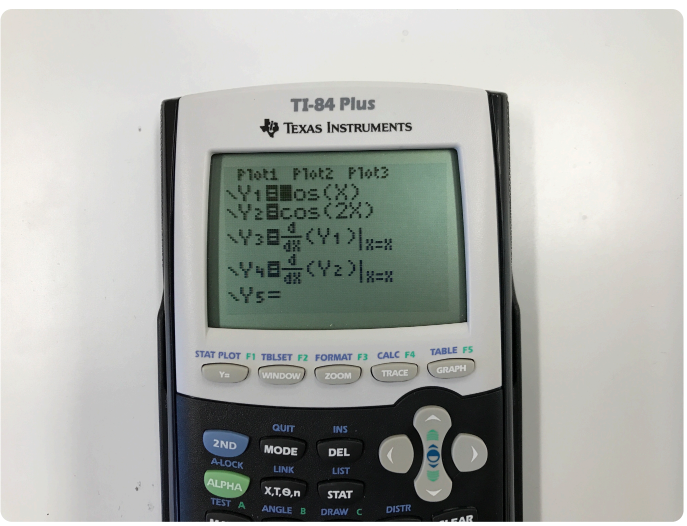
When you are done with your homework you should be able to…

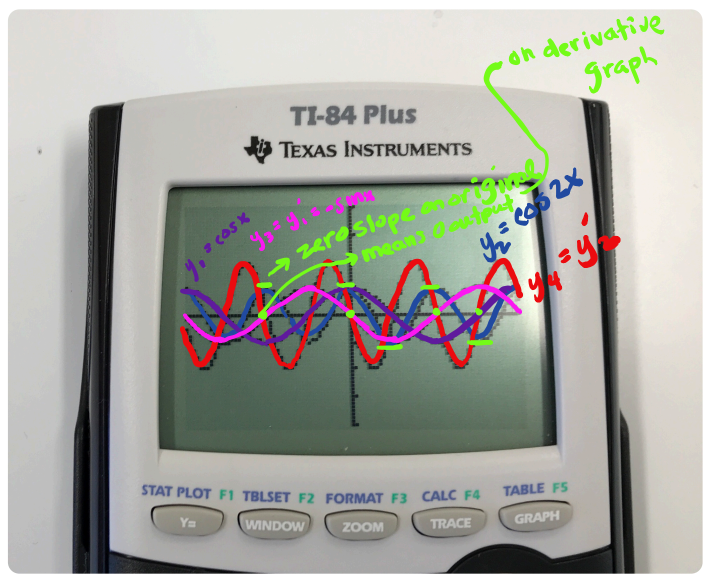
- π Find the derivative of a composite function using the Chain Rule.
- π Find the derivative of a function using the General Power Rule.
- π Simplify the derivative of a function using algebra.
- π Find the derivative of a trigonometric function using the Chain Rule.

Theorem: The Chain Rule

If $y=f(u)$ is a differentiable function of *u* and $u=g(x)$ is a differentiable function of *x*, then $y=f(g(x))$ is a differentiable function of *x* and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \text{ or } \frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x).$$





$h(x) = (f \circ g)(x)$

$h(x) = f[g(x)]$

$h(x) = \cos 2x$; Let $u=2x$

$f(u) = \cos u$

$u = 2x$

$\frac{d}{dx} h(x) = \frac{d}{dx}(\cos 2x)$

$h'(x) = -\sin(2x) \frac{d}{dx} 2x$

$h'(x) = (-\sin 2x)\cdot 2$

$h'(x) = -2\sin 2x$

$h'(x) = f'[g(x)]\, g'(x)$

$h'(x) = [f'(u)] \frac{du}{dx}$

Example 1: Use the methods learned in 2.2 and 2.3 to evaluate the derivative of the following functions. Then find the derivative using the Chain Rule.

pascal's Δ for expansion

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

a. $y=(2-x)^3 = (2-x)(2-x)(2-x)$

"old way"

$y=(2+(-x))^3$

$y = 1(2)^3(-x)^0 + 3(2)^2(-x)^1 + 3(2)^1(-x)^2 + 1(2)^0(-x)^3$

$\frac{d}{dx}y = \frac{d}{dx}(8-12x+6x^2-x^3)$

$y' = -12+12x-3x^2 \rightarrow y' = -3(x^2-4x+4) \rightarrow y' = -3(x-2)^2$

$y' = -3[(-1)(2-x)]^2$

$y' = -3(2-x)^2$

Chain Rule

$\frac{d}{dx}y = \frac{d}{dx}(2-x)^3$

$u=2-x \quad f(u)=u^3$

$\frac{du}{dx}=-1 \quad f'(u)=3u^2$

$\frac{dy}{dx} = 3(2-x)^2\frac{d}{dx}(2-x)$

$\frac{dy}{dx} = 3(2-x)^2(-1) \rightarrow \boxed{\frac{dy}{dx} = -3(2-x)^2}$

b. $f(x) = \sin 2x$

$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x)+f(x)g'(x)$

"old way"

$\frac{d}{dx}f(x) = \frac{d}{dx}2\sin x\cos x$

applied product rule

$f'(x) = 2\left[\left(\frac{d}{dx}\sin x\right)\cos x + (\sin x)\left(\frac{d}{dx}\cos x\right)\right]$

$f'(x) = 2[\cos x\cos x + \sin x(-\sin x)]$

$f'(x) = 2(\cos^2 x - \sin^2 x) \rightarrow f'(x) = 2\cos 2x$

Chain Rule

$\frac{d}{dx}f(x) = \frac{d}{dx}\sin 2x$

$f'(x) = \cos(2x)\left(\frac{d}{dx}2x\right)$

$f'(x) = (\cos 2x)\cdot 2$

$\boxed{f'(x) = 2\cos 2x}$

quotient rule: $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

c. $h(t) = \frac{\sqrt{t}}{\sqrt{t}-1} \rightarrow h(t) = \frac{t^{1/2}}{t^{1/2}-1}$

"old way"

$$h'(t) = \frac{\left(\frac{d}{dt}t^{1/2}\right)(t^{1/2}-1) - t^{1/2}\left[\frac{d}{dt}(t^{1/2}-1)\right]}{(t^{1/2}-1)^2}$$

$$h'(t) = \frac{\frac{1}{2}t^{-1/2}(t^{1/2}-1) - t^{1/2}\left(\frac{1}{2}t^{-1/2}\right)}{(t^{1/2}-1)^2}$$

$$h'(t) = \frac{\cancel{\frac{1}{2}t^0} - \frac{1}{2}t^{-1/2} - \cancel{\frac{1}{2}t^0}}{(t^{1/2}-1)^2}$$

$$\boxed{h'(t) = -\frac{1}{2t^{1/2}(t^{1/2}-1)^2}}$$

Chain Rule

$h(t) = \frac{t^{1/2}}{t^{1/2}-1}$ rewrite as a product

$\frac{d}{dt}h(t) = \frac{d}{dt}\left(t^{1/2}(t^{1/2}-1)^{-1}\right)$ apply product rule: $\frac{d}{dt}[f(t)g(t)] = f'(t)g(t) + f(t)g'(t)$

$$h'(t) = \left(\frac{d}{dt}t^{1/2}\right)(t^{1/2}-1)^{-1} + (t^{1/2})\left[\frac{d}{dt}(t^{1/2}-1)^{-1}\right]$$

$$h'(t) = \frac{1}{2}t^{-1/2}(t^{1/2}-1)^{-1} + t^{1/2}\left[-1(t^{1/2}-1)^{-2}\left[\frac{d}{dt}(t^{1/2}-1)\right]\right]$$

$$h'(t) = \frac{1}{2}t^{-1/2}(t^{1/2}-1)^{-1} - t^{1/2}(t^{1/2}-1)^{-2}\left(\frac{1}{2}t^{-1/2}\right)$$

zero out

$$h'(t) = \frac{1}{2}t^{-1/2}(t^{1/2}-1)^{-2}\left[(\cancel{t^{1/2}}-1)^1 - \cancel{t^{1/2}}\right]$$

$$\boxed{h'(t) = \frac{-1}{2t^{1/2}(t^{1/2}-1)^2}}$$

Theorem: The General Power Rule

If $y=\left[u(x)\right]^n$, where u is a differentiable function of x and n is a rational number, then

$$\frac{dy}{dx}=n\left[u(x)\right]^{n-1}\cdot\frac{du}{dx} \text{ or } \frac{d}{dx}\left[u^n\right]=nu^{n-1}u'.$$

Example 2: Complete the table.

$y=f(g(x))$	$u=g(x)$	$y=f(u)$
$y=\left(8x^2-3\right)^{25}$	$u=8x^2-3$	$y=u^{25}$
$y=\tan\frac{\pi x}{3}$	$u=\frac{\pi x}{3}$	$y=\tan u$
$y=\csc^2 x$	$u=\csc x$	$y=u^2$
$y=\frac{5}{\sqrt{x^2+6}}$	$u=x^2+6$	$y=\frac{5}{u^{1/2}}=5u^{-1/2}$

Example 3: Find the derivative of the following functions.

a. $y = \sec x$

$y' = \sec x \tan x$ Basic formula

b. $y = \sec 2x$

$y' = \sec(2x)\tan(2x)\left(\frac{d}{dx} 2x\right)$

$y' = (\sec 2x \tan 2x) \cdot 2$

$y' = 2\sec 2x \tan 2x$

chain rule

$u = 2x \quad f(u) = \sec u$

$\frac{du}{dx} = 2 \quad f'(u) = \sec u \tan u$

c. $y = \sec^2 x \rightarrow y = (\sec x)^2$

$y' = 2(\sec x)^1\left(\frac{d}{dx}\sec x\right)$

$y' = (2\sec x)(\sec x \tan x)$

$\boxed{y' = 2\sec^2 x \tan x}$

chain rule

$u = \sec x \quad f(u) = u^2$

$\frac{du}{dx} = \sec x \tan x \quad f'(u) = 2u$

d. $y = \sec x^2$

$y' = [\sec(x^2)\tan(x^2)]\left(\frac{d}{dx} x^2\right)$

$y' = (\sec x^2 \tan x^2)(2x)$

$\boxed{y' = 2x \sec x^2 \tan x^2}$

chain rule

$u = x^2 \quad f(u) = \sec u$

$\frac{du}{dx} = 2x \quad f'(u) = \sec u \tan u$

e. $y = x^5$

f. $y = \left(2x^3 - 5\right)^5$

g. $y = \sqrt{x}$

$y' = \frac{1}{2\sqrt{x}}$

h. $y = \sqrt{\cos x}$

$y' = \frac{1}{2\sqrt{\cos x}} \frac{d}{dx} \cos x$

$\boxed{y' = \frac{-\sin x}{2\sqrt{\cos x}}}$

i. $f(x) = x^2(2-x)^{2/3}$

d. $f(x)=\sqrt{\frac{1}{2x^3+15}} = \frac{\sqrt{1}}{\sqrt{2x^3+15}} = (2x^3+15)^{-1/2}$ Algebra

$f'(x) = -\frac{1}{2}(2x^3+15)^{-3/2}\left[\frac{d}{dx}(2x^3+15)\right]$ chain rule

$f'(x) = -\frac{1}{\cancel{2}}(2x^3+15)^{-3/2}(\cancel{6}^{\,3}x^2)$

$$\boxed{f'(x) = -\frac{3x^2}{(2x^3+15)^{3/2}}}$$

e. $\frac{d}{dx} h(x) = \frac{d}{dx}\left(x\sin^2 4x\right)$

$h'(x) = \left[\frac{d}{dx}(x)\right]\left[\sin^2 4x\right] + (x)\left[\frac{d}{dx}(\sin 4x)^2\right]$ product rule

$h'(x) = 1\sin^2 4x + x\underbrace{\left[2(\sin 4x)^1\right]\left[\frac{d}{dx}\sin 4x\right]}_{\text{chain rule}}$

$h'(x) = \sin^2 4x + 2x\sin 4x\cos(4x)\left[\frac{d}{dx}4x\right]$

$h'(x) = \sin^2 4x + (2x\sin 4x\cos 4x)(4)$

$\boxed{h'(x) = \sin^2 4x + 8x\sin 4x\cos 4x}$

f. Find the equation of the tangent line at $t=1$ for the function

$$s(t)=\left(9-t^2\right)^{2/3}.$$

$y-y_1=m(t-t_1)$

$t_1=1$

need $s(t_1)=y_1$

$s(1)=(9-(1)^2)^{2/3}=(\sqrt[3]{8})^2=4$

$(t_1,y_1)=(1,4)$

need m:

$m=s'(t)=\frac{2}{3}(9-t^2)^{-1/3}\left[\frac{d}{dt}(9-t^2)\right]$ chain rule

$s'(t)=\frac{2}{3}(9-t^2)^{-1/3}(-2t)$

$s'(1)=\frac{2}{3}(9-(1)^2)^{-1/3}(-2(1))$

$s'(1)=\frac{2}{3}\cdot\frac{1}{\sqrt[3]{8}}\cdot -2$

$s'(1)=-\frac{2}{3}$

construct eq. of line:

$y-y_1=m(t-t_1)$

$\boxed{y-4=-\frac{2}{3}(t-1)}$

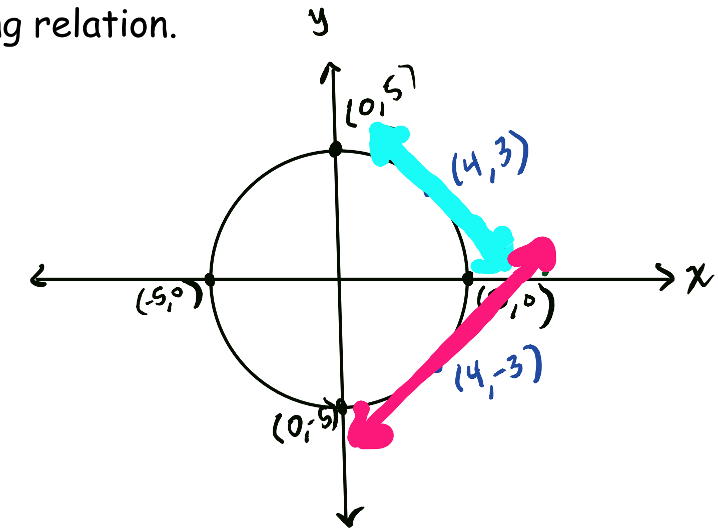
2.5: Implicit Differentiation

When you are done with your homework you should be able to…

- π Distinguish between functions written in implicit form and explicit form
- π Use implicit differentiation to find the derivative of a function

Warm-up: Find the derivative of the following relation.

$x^2 + y^2 = 25$



$y^2 = 25 - x^2$

$y = \pm\sqrt{25-x^2}$

$\frac{d}{dx} y_1 = \frac{d}{dx}(25-x^2)^{1/2}$ or $y_2 = -(25-x^2)^{1/2}$

$\frac{dy_1}{dx} = \frac{1}{2}(25-x^2)^{-1/2}\left(\frac{d}{dx}(25-x^2)\right)$

$\frac{dy_1}{dx} = \frac{1}{\cancel{2}}(25-x^2)^{-1/2}(-\cancel{2}x)$

$\frac{dy_1}{dx} = \frac{-x}{\sqrt{25-x^2}}$ and $\frac{dy_2}{dx} = \frac{x}{\sqrt{25-x^2}}$ opposite sign

$\boxed{\frac{dy}{dx} = \pm\frac{x}{\sqrt{25-x^2}}}$

At $x = 4$, $y = \pm 3$

and $\frac{dy}{dx} = \pm\frac{4}{3}$

$y - y_1 = m(x - x_1)$

At $(4,3)$:

$y - 3 = -\frac{4}{3}(x-4)$

At $(4,-3)$:

$y-(-3) = \frac{4}{3}(x-4)$

$y+3 = \frac{4}{3}(x-4)$

IMPLICIT AND EXPLICIT RELATIONS

Up to this point, we have typically seen functions expressed in explicit form.

$y = 2x^2 - \sqrt{x}$

$\sqrt{x} + y = 2x^2$

$x^2 - 3x + 4y^2 + 3y - 29 = 0$

hard to isolate y

Some relations or functions are only implied by an equation.

$x^2+y^2=25$ or $\sin(x+y)=\sin(xy)-5$

When you were differentiating the warm-up problem, you were able to explicitly write y as a function of x, using 2 equations.

$x^2+y^2=25$

$y=\sqrt{25-x^2}$ or $y=-\sqrt{25-x^2}$

Oftentimes, it is difficult to write y as a function of x explicitly. In order to differentiate we must use implicit differentiation. To understand how to find $\frac{dy}{dx}$ implicitly, you must know which variable you are differentiating with respect to. You must use the chain rule on any variable which is different than the one you are differentiating with respect to.

y' is $\frac{dy}{dx}$ or $\frac{dy}{d[\ ]}$

Example 1: Find the derivative of the following functions with respect to x. Simplify your result to a single rational expression with positive exponents.

a. $\frac{d}{dx}y=\frac{d}{dx}x^5$

$\boxed{\frac{dy}{dx}=5x^4}$

matched b. $\frac{d}{dx}x=\frac{d}{dx}y^5$ doesn't match → implicit

$1=5(y)^4\left(\frac{d}{dx}y\right)$

$1=5y^4\frac{dy}{dx}$

$\boxed{\frac{dy}{dx}=\frac{1}{5y^4}}$

c. $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25$

Recall...

$y = \pm\sqrt{25 - x^2}$

$\frac{d}{dx}x^2 + \underbrace{\frac{d}{dx}y^2} = 0$

$2x + \underbrace{2(y)^1 \frac{d}{dx}(y)} = 0$

$2x + \underbrace{2y\left(\frac{dy}{dx}\right)} = 0$

$2y\left(\frac{dy}{dx}\right) = -2x$

$\boxed{\frac{dy}{dx} = -\frac{x}{y}}$

or $\boxed{\frac{dy}{dx} = \pm\frac{x}{\sqrt{25-x^2}}}$

GUIDELINES FOR IMPLICIT DIFFERENTIATION (WITH RESPECT TO *x*)

1. Differentiate both sides of the equation with respect to *x*.
2. Collect all terms involving $\frac{dy}{dx}$ on the left side of the equation and move all other terms to the right side of the equation.
3. Factor $\frac{dy}{dx}$ out of the left side of the equation.
4. Isolate $\frac{dy}{dx}$.

Example 2: Find the derivative of the following functions with respect to *x*. Simplify your result to a single rational expression with positive exponents.

a. $4\cos x \sin y = 1$

$$\frac{d}{dx}(\cos x \sin y) = \frac{d}{dx}\frac{1}{4}$$

$$\left(\frac{d}{dx}\cos x\right)\sin y + \cos x\left(\frac{d}{dx}\sin y\right) = 0$$ product rule

$$-\sin x \sin y + \cos x\left[\cos(y)\frac{d}{dx}(y)\right] = 0$$ chain rule

$$-\sin x \sin y + \cos x \cos y\left(\frac{dy}{dx}\right) = 0$$

$$\cos x \cos y\left(\frac{dy}{dx}\right) = \sin x \sin y$$

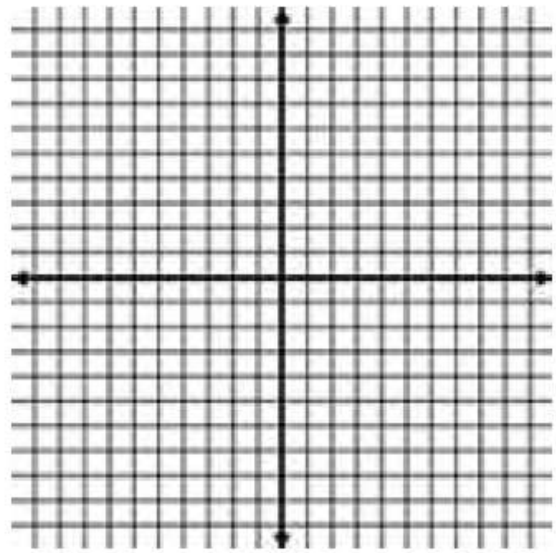
$$\frac{dy}{dx} = \frac{\sin x \sin y}{\cos x \cos y}$$

$$\boxed{\frac{dy}{dx} = \tan x \tan y}$$

b. $x^2y + y^2x = -2$

c. $x = \csc\dfrac{1}{y}$

Example 3: Use implicit differentiation to find an equation of the tangent line to the hyperbola $\frac{x^2}{4}-\frac{y^2}{9}=1$ at $x=4$. Sketch the graphs of the hyperbola and the tangent line at $x=4$.



Example 4: Find the points at which the graph of the equation $4x^2 + y^2 - 8x + 4y + 4 = 0$ has a horizontal tangent line.

$$\frac{d}{dx}\left(4x^2 + y^2 - 8x + 4y + 4\right) = \frac{d}{dx}(0)$$

$$\frac{d}{dx}4x^2 + \frac{d}{dx}y^2 - 8\frac{d}{dx}x + 4\frac{d}{dx}y + \frac{d}{dx}4 = 0$$

$$8x + 2(y)^1\frac{d}{dx}y - 8(1) + 4\frac{dy}{dx} + 0 = 0$$

$$8x + 2y\left(\frac{dy}{dx}\right) - 8 + 4\left(\frac{dy}{dx}\right) = 0$$

$$2y\left(\frac{dy}{dx}\right) + 4\left(\frac{dy}{dx}\right) = 8 - 8x$$

$$\frac{dy}{dx}\left(2y + 4\right) = 8 - 8x$$

$$\frac{dy}{dx} = \frac{8(1-x)}{2(y+2)}$$

Slope of graph: $\frac{dy}{dx} = \frac{4(1-x)}{(y+2)}$

Where horizontal tangents are at:

$$(y+2)\cdot 0 = \frac{4(1-x)}{\cancel{y+2}} \cdot \cancel{(y+2)}$$

$$0 = 4(1-x)$$

$$0 = 1-x$$

$$x = 1$$

Find y-coordinates

At $x=1$: $4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$

$$y^2 + 4y = 0$$

At the points $(1,0)$ and $(1,-4)$ this graph has horizontal tangent lines.

$$y(y+4) = 0$$

$y = 0$ or $y = y+4$

$y = -4$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''$$

Example 5: Find $\dfrac{d^2y}{dx^2}$ in terms of x and y.

$$\frac{d}{dx}(1-xy) = \frac{d}{dx}(x-y)$$

$$\frac{d}{dx}1 - \frac{d}{dx}xy = \frac{d}{dx}x - \frac{d}{dx}y$$

$$0 - \left[\left(\frac{d}{dx}x\right)y + x\left(\frac{d}{dx}y\right)\right] = 1 - \frac{dy}{dx}$$

$$-\left[1y + x\frac{dy}{dx}\right] = 1 - \frac{dy}{dx}$$

$$-y - x\frac{dy}{dx} + \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} - x\frac{dy}{dx} = 1 + y$$

$$\frac{dy}{dx}(1-x) = 1+y$$

$$\frac{dy}{dx} = \frac{1+y}{1-x}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1+y}{1-x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{d}{dx}(1+y)\right)(1-x) - (1+y)\left(\frac{d}{dx}(1-x)\right)}{(1-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\left(0+\frac{dy}{dx}\right)(1-x) - (1+y)\overset{(-1)}{(0-1)}}{(1-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy}{dx}(1-x) + (1+y)}{(1-x)^2}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{1+y}{\cancel{1-x}}\overset{1}{\cancel{(1-x)}} + (1+y)}{(1-x)^2}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{2(1+y)}{(1-x)^2}}$$

2.6: Related Rates

When you are done with your homework you should be able to…

- π Find a related rate
- π Use related rates to solve real-life problems

Warm-up 1: Find the derivative of V with respect to t.

$$\frac{d}{dt} V \overset{\frac{d}{dt}}{=} r^2 h$$

$$\frac{dV}{dt} = \left(\frac{d}{dt} r^2\right) h + r^2 \left(\frac{d}{dt} h\right)$$ product rule

$$\frac{dV}{dt} = 2(r)^1 \frac{d}{dt}(r) h + r^2 \frac{dh}{dt}$$

$$\boxed{\frac{dV}{dt} = 2r \frac{dr}{dt} \cdot h + r^2 \frac{dh}{dt}}$$

Warm-up 2:

Find the volume of a cone with a radius of 24 inches and a height of 10 inches. Round to the nearest hundredth.

$$V = \frac{\pi r^2 h}{3}$$

$$V = \frac{(24)^2 (10)}{3}$$

$$V = 1920\pi \text{ in}^3$$

FINDING RELATED RATES

We use the ___chain___ rule to ___implicitly___ find the rates of change of two or more related variables that are changing with respect to ___time___.

Some common formulas used in this section:

- Volume of a…
  - Sphere: $V = \frac{4}{3}\pi r^3$
  - Right Circular Cylinder: $V = \pi r^2 h$
  - Right Circular Cone: $V = \frac{\pi r^2 h}{3}$
  - Right Rectangular Prism: $V = lwh$, $SA = 2lw + 2lh + 2wh$
  - Right Rectangular Pyramid: $V = \frac{lwh}{3}$
- Pythagorean Theorem: $a^2 + b^2 = c^2$

GUIDELINES FOR SOLVING RELATED-RATE PROBLEMS

1. Analyze: Identify all **given** quantities and quantities **to be determined**. Make a sketch and label the quantities.

2. Related Variables Equation: Write an equation involving the variables whose rates of change either are given or are to be determined.

3. Find the Related Rate: Using the **Chain Rule**, implicitly differentiate both sides of the equation **with respect to time *t***.

4. Find Desired Rate: **After** completing step 3, substitute into the **resulting equation** all known values for the variables and their rates of change. Then solve for the required rate of change.

5. Conclusion: Write your conclusion in **words**.

Example 1: Find the **rate of change of the distance** between the origin and a **moving point** on the graph of $y = \sin x$ if $\frac{dx}{dt} = 2\text{cm/sec}$.

1. Analyze:

2. Related Variables Equation:

3. Find Related Rate:

4. Find Desired Rate:

5. Conclusion:

Example 2: Find the **rate of change of the volume** of a cone **if dr/dt is 2 inches per minute and h = 3r when r = 6 inches**. Round to the nearest hundredth. How is this problem different than the warm-up problem?

1. Analyze:

$$V = \frac{\pi r^2 h}{3}$$

$\frac{dr}{dt} = 2$ in/min exactly when $r = 6$ and $h = 3r$ ~~$= 3 \cdot 6 = 18$~~

Don't need since we're rewriting the V

we want to find $\frac{dV}{dt}$

$$V = \frac{\pi r^2 (\cancel{3}r)}{\cancel{3}} = \pi r^3$$

2. Related Variables Equation:

$$V = \pi r^3$$

3. Find Related Rate:

$$\frac{d}{dt} V = \frac{d}{dt} \pi r^3$$

$$\frac{dV}{dt} = \pi \cdot 3 (r)^2 \frac{d}{dt} (r)$$

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt}$$

4. Find Desired Rate:

$$\frac{dV}{dt} = 3\pi r^2 \frac{dr}{dt} \rightarrow \frac{dV}{dt} = 3\pi (6\text{in})^2 \left(\frac{2\text{in}}{\text{min}}\right) \rightarrow \frac{dV}{dt} = 216\pi \text{ in}^3/\text{min}$$

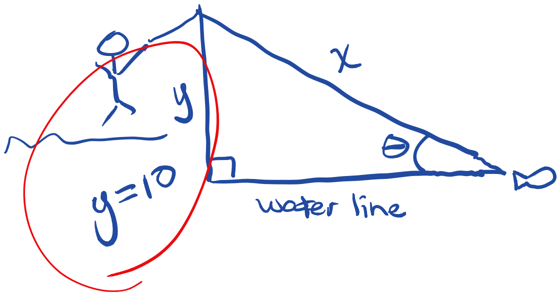
5. Conclusion:

The instantaneous rate of change of the volume is 216π inches cubed per minute when the radius is 6 inches.

Example 3: Angle of Elevation. A fish is reeled in at a rate of 1 foot per second from a point 10 feet above (y) the water. At what rate is the angle between the line and the water changing when there is a total of 25 feet of line out?

1. Analyze:

$\frac{dx}{dt} = -1$ ft/sec



$\sin\theta = \frac{y}{x} = \frac{10}{x}$

We want $\frac{d\theta}{dt}$ exactly when $x = 25$ ft

2. Related Variables Equation:

$\sin\theta = \frac{y}{x} \rightarrow \sin\theta = \frac{10}{x} = 10x^{-1}$

$a^2 + b^2 = c^2$

$(10)^2 + b^2 = x^2$

$b^2 = x^2 - 100$

$b = \sqrt{x^2 - 100}$

3. Find Related Rate:

$\frac{d}{dt}\sin\theta = \frac{d}{dt}10x^{-1}$

$\cos(\theta)\frac{d}{dt}(\theta) = 10\left(-1(x)^{-2}\frac{d}{dt}x\right)$

$\cos\theta\frac{d\theta}{dt} = -10x^{-2}\frac{dx}{dt}$

$\frac{d\theta}{dt} = \frac{-10(dx/dt)}{x^2\cos\theta}$

4. Find Desired Rate:

$\frac{d\theta}{dt} = \frac{-10(-1\text{ ft/sec})}{(25)^2\left[\frac{\sqrt{x^2-100}}{x}\right]} \rightarrow \frac{d\theta}{dt} = \frac{+10}{625\frac{\sqrt{25^2-100}}{25}} \rightarrow \frac{d\theta}{dt} = \frac{+10}{25\sqrt{525}} \approx 0.02$ rad/sec

5. Conclusion:

The rate of change of the angle with respect to time when there's 25 ft of line is approximately 0.02 radians/second.

Example 4: Consider the following situation:

A container, in the shape of an inverted right circular cone, has a radius of 5 inches at the top and a height of 7 inches. At the instant when the water in the container is 6 inches deep, the surface level is falling at the rate of -1.3 in/s. Find the rate at which the water is being drained.

1. Analyze:

2. Related Variables Equation:

3. Find Related Rate:

4. Find Desired Rate:

5. Conclusion:

Example 5: Shannon is trying to hang her Christmas lights. She has a ladder that is 22 feet long. It is leaning against a wall of her house. Since Shannon did not secure the ladder, it is moving away from the wall at a rate of 1.5 feet per second. How fast is the top of the ladder moving down the wall when its base is 10 feet from the wall?

1. Analyze:

2. Related Variables Equation:

3. Find Related Rate:

4. Find Desired Rate:

5. Conclusion:

3.1: Extrema on an Interval

When you are done with your homework you should be able to…

- π Understand the definition of a function on an interval
- π Understand the definition of relative extrema of a function on an open interval
- π Find extrema on a closed interval

Warm-up: Determine the point(s) at which the graph of $f(x)=2x^2-8x+5$ has a horizontal tangent.

Find the zeros of the slope function

$f(x)=2x^2-8x+5$

$f'(x)=4x-8$

$0=4(x-2)$

$\cancel{0=4}$ or $0=x-2$

extraneous

$x=2$

Find the y-coordinate corresponding to $x=2$

$f(2)=2(2)^2-8(2)+5$

$f(2)=8-16+5$

$f(2)=-3$

At $(2,-3)$ f has a horizontal tangent.

EXTREMA OF A FUNCTION

In calculus, much effort is devoted to determining the behavior of a function f on an interval I. Does f have a maximum value or minimum value on I? Where is f increasing? Where is f decreasing? In this chapter, you will learn how differentiation can be used to answer these questions.

On what interval

DEFINITION OF EXTREMA

Let f be defined on an open interval I containing c.

1. $f(c)$ is the **minimum of** f **on** I if $f(c) \le f(x)$ for all x in I.

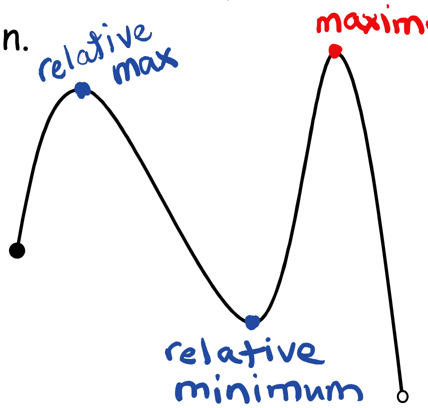
2. $f(c)$ is the **maximum of** f **on** I if $f(c) \ge f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the **extreme values**, or **extrema** (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the **absolute minimum** and **absolute maximum**, or the **global minimum** and **global maximum**, on the interval.

Let's check out the functions below. Identify the maximum and minimum of each function.

maximum = absolute max = global max

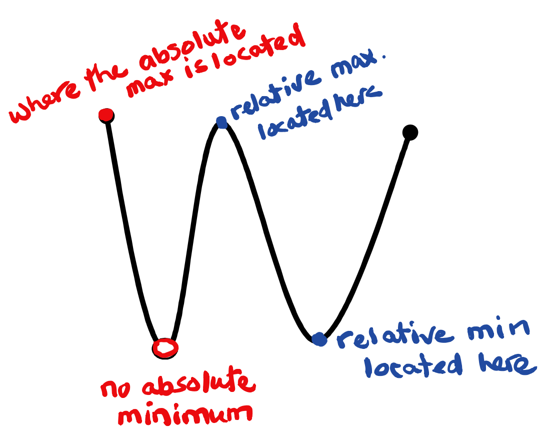
a.



no minimum = absolute minimum = global min

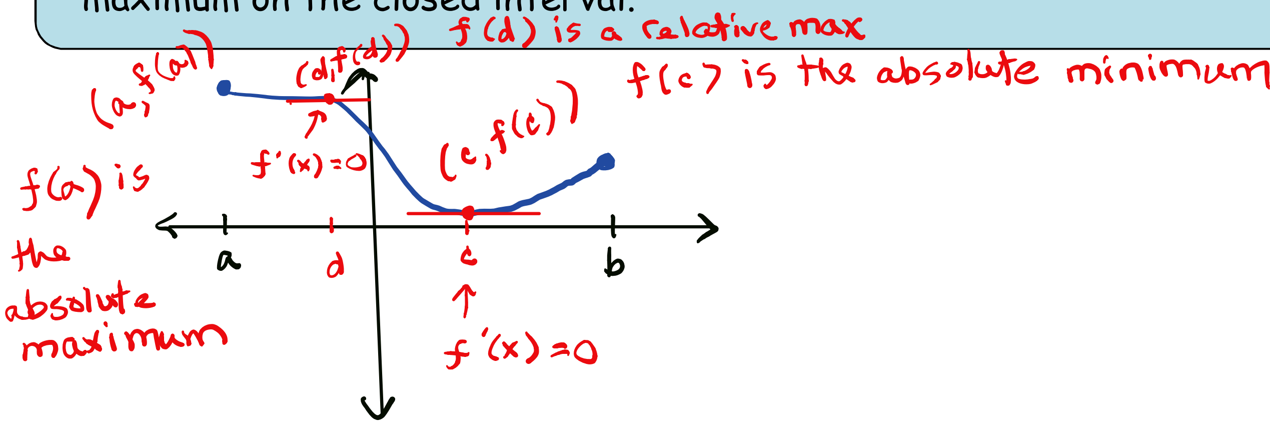
The ordered pairs give you possible locations of extrema; the y-coordinates would represent the possible max or min values.

b.



THEOREM: THE EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a,b]$, then f has both a minimum and a maximum on the closed interval.



DEFINITION OF RELATIVE EXTREMA

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a **relative maximum of** f, or you can say that f has a **relative maximum at** $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a **relative minimum of** f, or you can say that f has a **relative minimum at** $(c, f(c))$.

Example 1: Find the value of the derivative at the extremum $(0,1)$ for the function $f(x) = \cos\frac{\pi x}{2}$.

$f'(x) = -\frac{\pi}{2}\sin\frac{\pi x}{2}$

$f'(0) = -\frac{\pi}{2}\cdot\sin\frac{\pi(0)}{2}$

$f'(0) = 0$

crunching it out

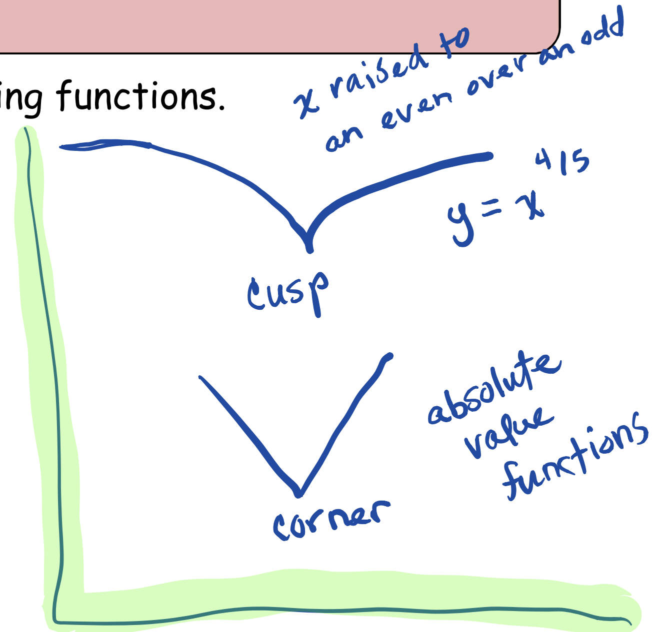
so $f'(0) = 0$

mad calculus knowledge

DEFINITION OF A CRITICAL NUMBER

Let f be defined at c. If $f'(c)=0$ **OR** if f is not differentiable at c, then c is a **critical number**.

Example 2: Find any critical numbers of the following functions.



a. $f(x)=2x^2-8x+5$

$f'(x) = 4x-8$

$0 = 4(x-2)$

$0 \cancel{\neq} 4$ or $0 = x-2$

$x = 2$

$\boxed{c = 2}$

b. $g(x)=\sin^2 x-\sin x,\ [0,2\pi)$

$g'(x) = 2(\sin x)'\cos x - \cos x$

$g'(x) = \cos x\,(2\sin x - 1)$

$0 = \cos x\,(2\sin x - 1)$

$\cos x = 0$ or $2\sin x - 1 = 0$

$\sin x = \frac{1}{2}$

$x = \frac{\pi}{2}, \frac{3\pi}{2}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$

$$\boxed{c_1 = \frac{\pi}{2},\ c_2 = \frac{3\pi}{2},\ c_3 = \frac{\pi}{6},\ c_4 = \frac{5\pi}{6}}$$

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)=\frac{f'(x)g(x)-f(x)g'(x)}{[g(x)]^2}$$

c. $s(t)=\dfrac{3t}{t^2-4}$

$$s'(t)=\frac{3(t^2-4)-3t(2t)}{(t^2-4)^2}$$

$$s'(t)=\frac{3t^2-12-6t^2}{(t^2-4)^2}$$

$$s'(t)=\frac{-3t^2-12}{(t^2-4)^2}$$

$$0=-3t^2-12$$

$$3t^2=-12$$

$$t^2=-4$$

$$t=\pm 2i$$

So no critical numbers from the zeros of the 1st derivative.

s is differentiable on its domain so no critical numbers from that.

$\boxed{\text{No critical numbers.}}$

At $t^2-4=0 \rightarrow t=\pm 2$ there are vertical asymptotes, so these values would be part of a sign chart for testing for ↑ and ↓ intervals.

THEOREM: RELATIVE EXTREMA OCCUR ONLY AT CRITICAL NUMBERS

If f has a relative minimum or relative maximum at $x=c$, then c is a critical number of f.

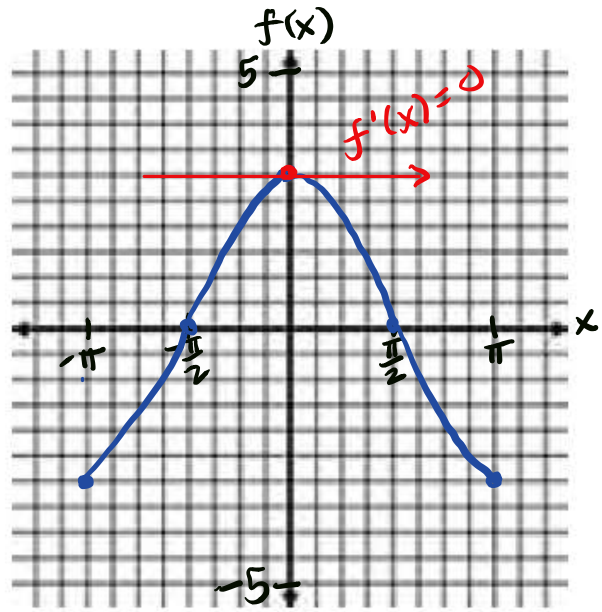
GUIDELINES FOR FINDING EXTREMA (AKA ABSOLUTE EXTREMA OR GLOBAL EXTREMA) ON A CLOSED INTERVAL

To find the extrema of a continuous function f on a closed interval $[a,b]$, use the following steps.

1. Find the critical numbers of f in (a,b).
2. Evaluate f at each critical number in (a,b).
3. Evaluate f at each endpoint of $[a,b]$.
4. The least of these numbers is the minimum. The greatest is the maximum.

Example 3: Locate the absolute extrema of the following functions function on the closed interval.

a. $f(x) = 3\cos x,\ [-\pi, \pi]$



1. Find critical numbers on $(-\pi, \pi)$.

$f'(x) = -3\sin x$

$0 = -3\sin x$

$0 = \sin x$

$x = 0$

$\underline{c = 0}$

2. Evaluate $f(0)$ = 3.

$f(0) = 3\cos 0 = 3(1) = 3$

3. Evaluate the function at the end points.

$f(-\pi) = 3\cos(-\pi) = 3\cos\pi = 3(-1) = -3$

$f(\pi) = 3\cos\pi = -3$

$\cos(-\theta) = \cos\theta$
$\sin(-\theta) = -\sin\theta$

4. Conclusion.

Absolute minimum is -3.
Absolute maximum is 3.

b. $g(t)=\dfrac{t}{t-2}$, $[3,5]$

1. Find critical numbers on $(3,5)$.

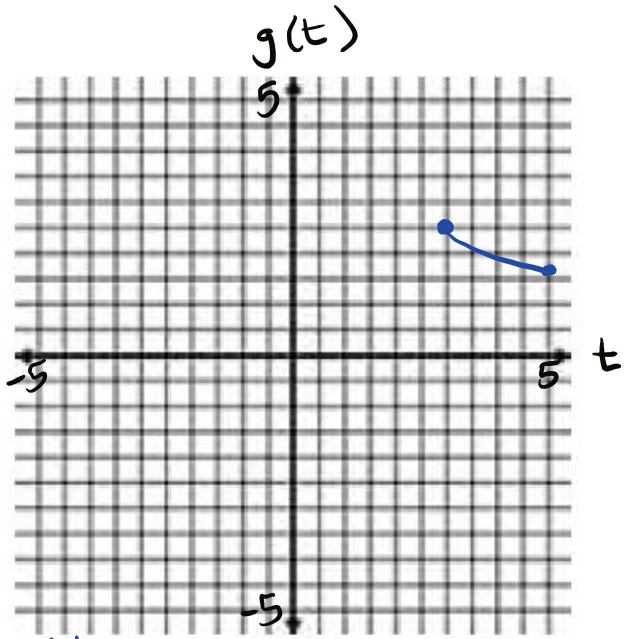
$$g'(t)=\frac{\left[\frac{d}{dt}(t)\right](t-2)-t\left[\frac{d}{dt}(t-2)\right]}{(t-2)^2}$$

$$g'(t)=\frac{1(t-2)-t(1)}{(t-2)^2}$$

$$g'(t)=\frac{t-2-t}{(t-2)^2}$$

$$g'(t)=\frac{-2}{(t-2)^2} \rightarrow 0=\frac{-2}{(t-2)^2} \rightarrow \overset{\text{impossible}}{0=-2} \rightarrow \text{no zeros of the 1st deriv}$$

$\boxed{\text{no critical numbers}}$



~~2~~. Evaluate _______ = _______.

no critical #'s

3. Evaluate the function at the endpoints.

$$g(3)=\frac{3}{3-2}=3$$

$$g(5)=\frac{5}{5-2}=\frac{5}{3}$$

4. Conclusion.

$\boxed{\begin{array}{l}\text{Absolute max is } 3\text{; located at } (3,3)\\ \text{Absolute min is } \frac{5}{3}\text{; located at } (5,5/3)\end{array}}$

→ at $t=3$, the max is 3

→ at $t=5$, the max is $\frac{5}{3}$

Example 4: A retailer has determined that the cost C of ordering and storing x units of a product is $C = 2x + \frac{300,000}{x},\ 1 \le x \le 300.$ The delivery truck can bring at most 300 units per order.

$\in$ = is an element of

a. Find the order size that will minimize cost.

$C(x) = 2x + 300000x^{-1}$

$C'(x) = 2 - 300000x^{-2}$

$C'(x) = 2 - \frac{300000}{x^2}$

$0 = 2 - \frac{300000}{x^2}$

$\cancel{x^2}\,\frac{300000}{\cancel{x^2}} = 2 \cdot x^2$

$300000 = 2x^2$

$150000 = x^2$

$\pm\sqrt{150000} = x$

$x = \pm 387 \notin 1 \le x \le 300$

So no critical numbers.

Evaluate endpoints:

$C(1) = 2(1) + \frac{300000}{1} = 300002$

$C(300) = 2(300) + \frac{3000\cancel{00}}{3\cancel{00}} = 1600$

An order size of 300 units will minimize cost.

b. Could the cost be decreased if the truck were replaced with one that could bring at most 400 units? Explain.

C.N. $= 387 \in 1 \le x \le 400$

$C(387) = 2(387) + \frac{300000}{387} \approx 1549$

$C(400) = 2(400) + \frac{300000}{400} \approx 1550$

$\boxed{\text{Yes. 387 units will minimize the cost.}}$

3.2: Rolle's Theorem and the Mean Value Theorem

When you are done with your homework you should be able to…

- π Understand and use Rolle's Theorem
- π Understand and use the Mean Value Theorem

Warm-up: Locate the global extrema of the function $f(x)=x^3-12x$ on the closed interval $[0,4]$.

$\boxed{1}$

$f'(x) = 3x^2 - 12$

$0 = 3x^2 - 12$

$12 = 3x^2$

$4 = x^2$

$\pm\sqrt{4} = x$

$\pm 2 = x$

$c = 2$

$-2 \notin (0,4)$

② $f(2) = (2)^3 - 12(2)$

$f(2) = -16$

③ $f(0) = (0)^3 - 12(0) = 0$

$f(4) = (4)^3 - 12(4) = 16$

④ Global min is -16; located at $(2,-16)$

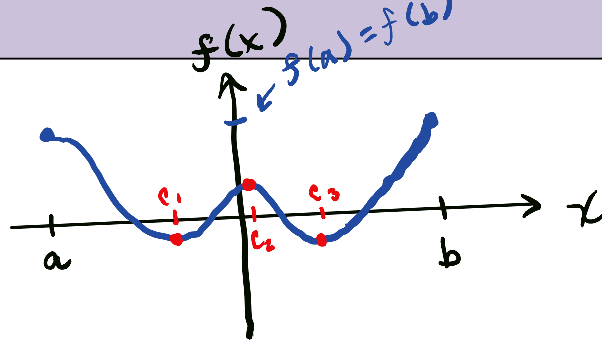
Global max is 16; located at $(4,16)$

ROLLE'S THEOREM

The ___Extreme___ Value Theorem states that a continuous function on a closed interval ___$[a,b]$___ must have both a ___minimum___ and a ___maximum___. Both of these values, however, can occur at the ___endpoints___. Rolle's Theorem gives conditions that guarantee the existence of an extreme value in the ___interior___ of a closed interval.

THEOREM: ROLLE'S THEOREM

Let f be continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b). If $f(a)=f(b)$ then there is at least one number c in (a,b) such that $f'(c)=0$.



Example 1: Determine whether Rolle's Theorem can be applied to *f* on the closed interval $[a,b]$. If Rolle's Theorem can be applied, find all values of *c* in the open interval (a,b) such that $f'(c)=0$.

conditions

a. $f(x)=\cos 2x,\ [-\pi,\pi]$

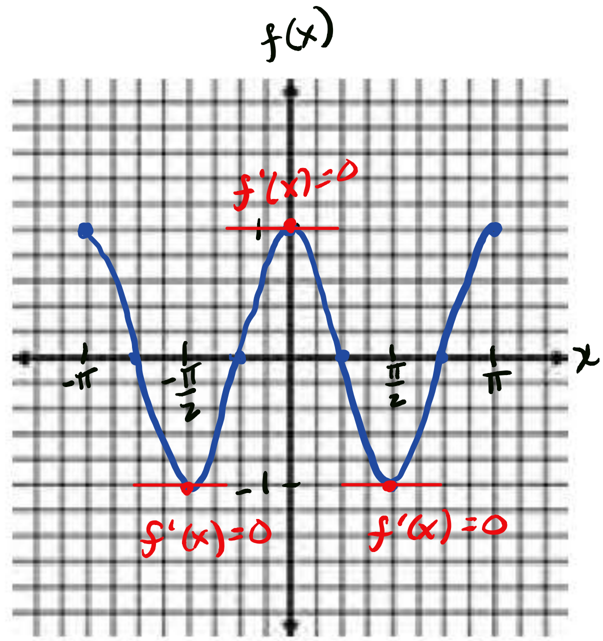
✓ $f(-\pi) = \cos(-2\pi) = \cos(2\pi) = 1$

$f(\pi) = \cos 2\pi = 1$

✓ f is continuous on $[-\pi,\pi]$.

✓ f is differentiable on $(-\pi,\pi)$.

So Rolle's Theorem applies.



Apply it!

$f'(x) = -2\sin 2x$

$0 = -2\sin 2x$, $-2\pi < 2x < 2\pi$

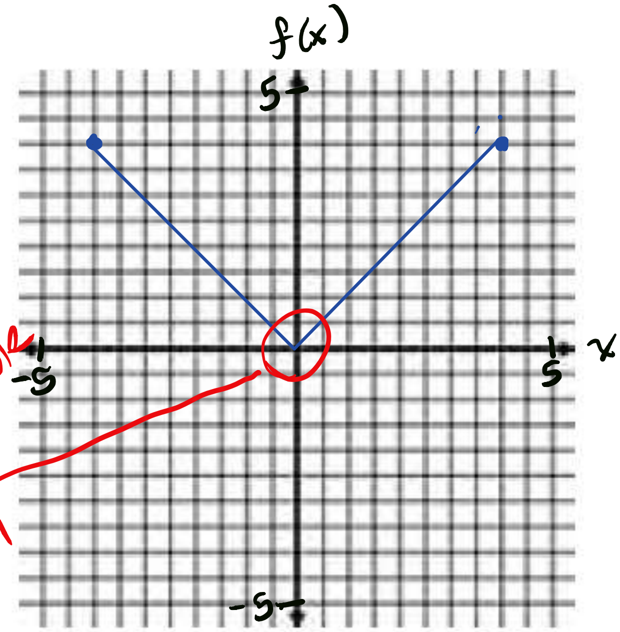
$0 = \sin 2x$

$2x = -\pi, 0, \pi$

$x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$

$\boxed{c_1 = -\frac{\pi}{2},\ c_2 = 0,\ c_3 = \frac{\pi}{2}}$

b. $f(x) = |x|,\ [-4, 4]$



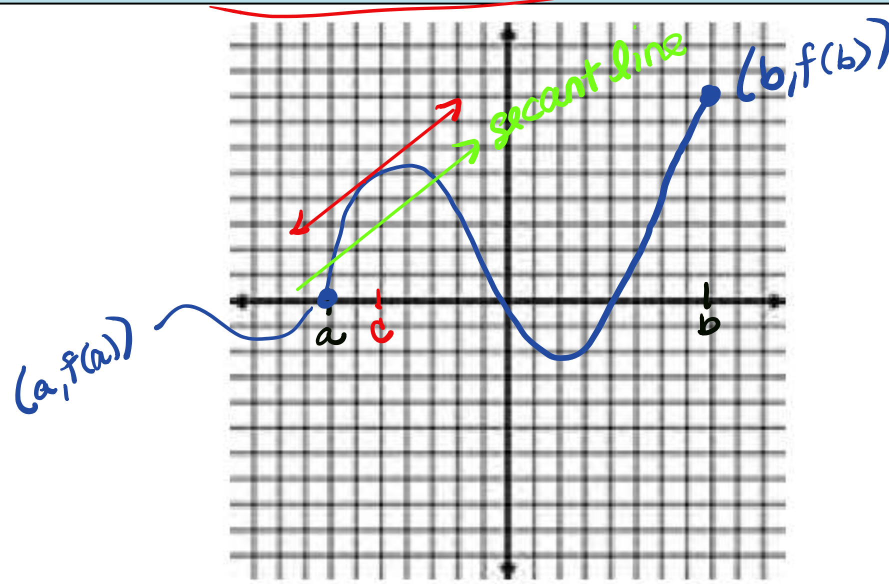
f is not differentiable at x=0, so Rolle's Theorem cannot be applied

THEOREM: THE MEAN VALUE THEOREM (MVT)

If f is continuous on the closed interval $[a,b]$, and differentiable on the open interval (a,b), then there exists a number c in (a,b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

instantaneous rate of change is equal to the average rate of change



Example 2: Consider the function $f(x)=\sqrt{x}$ on the closed interval from $[1,9]$.

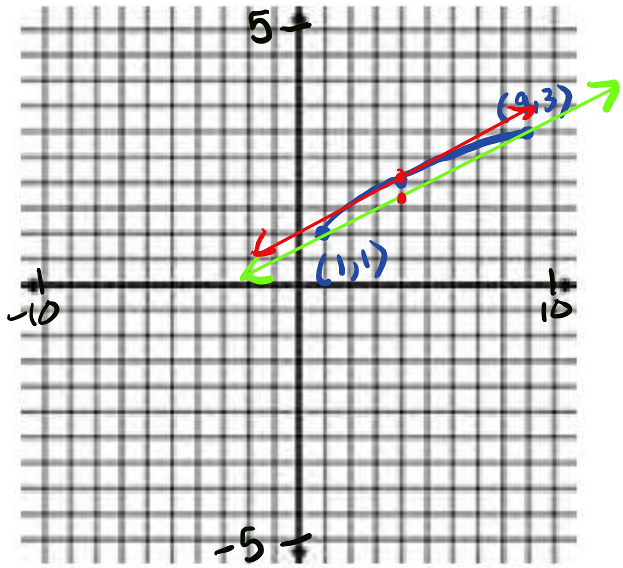
a. Graph the function on the given interval.

conditions

√ f is continous on [1,9].

√ f is differentiable (1,9)

So the MVT can be applied.



b. Find and graph the secant line through the endpoints on the same coordinate plane.

$$m_{sec}=\frac{f(b)-f(a)}{b-a}$$

$$m_{sec}=\frac{3-1}{9-1}=\frac{2}{8}=\frac{1}{4}$$

(1,1) is a point on the graph

$$\boxed{y-1=\frac{1}{4}(x-1)}$$

c. Find and graph any tangent lines to the graph of f that are parallel to the secant line.

parallel lines have the same slope.

$$f'(x)=\frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{x}}=\frac{1}{4}$$

$$4=2\sqrt{x}$$

$$(2)^2=(\sqrt{x})^2$$

$$4=x$$

must use (4,2)

$$\boxed{y-2=\frac{1}{4}(x-4)}$$

3.3: Increasing and Decreasing Function Intervals and the First Derivative Test

When you are done with your homework you should be able to…

- π Determine intervals on which a function is increasing or decreasing
- π Apply the First Derivative Test to find relative extrema of a function

Warm-up: Find the equation of the line tangent to the function $f(x) = \tan x$ at $x = \frac{3\pi}{4}$.

$y - y_1 = m(x - x_1)$; $m = f'(x)$

① $f'(x) = \sec^2 x$

$f'\left(\frac{3\pi}{4}\right) = \left(\sec\frac{3\pi}{4}\right)^2$

$f'(3\pi/4) = \left(-\frac{2}{\sqrt{2}}\right)^2$

$f'(3\pi/4) = 2 = m$

② $f(3\pi/4) = \tan(3\pi/4) = -1$

③ $y - (-1) = 2(x - 3\pi/4)$

$\boxed{y + 1 = 2(x - 3\pi/4)}$

INCREASING AND DECREASING FUNCTION INTERVALS

A function is __increasing__ if, as *x* moves to the right, its graph moves up, and is decreasing if its graph moves __down__. A positive derivative implies that the function is __increasing__ and a __negative__ derivative implies that the function is decreasing.

DEFINITION OF INCREASING AND DECREASING FUNCTIONS

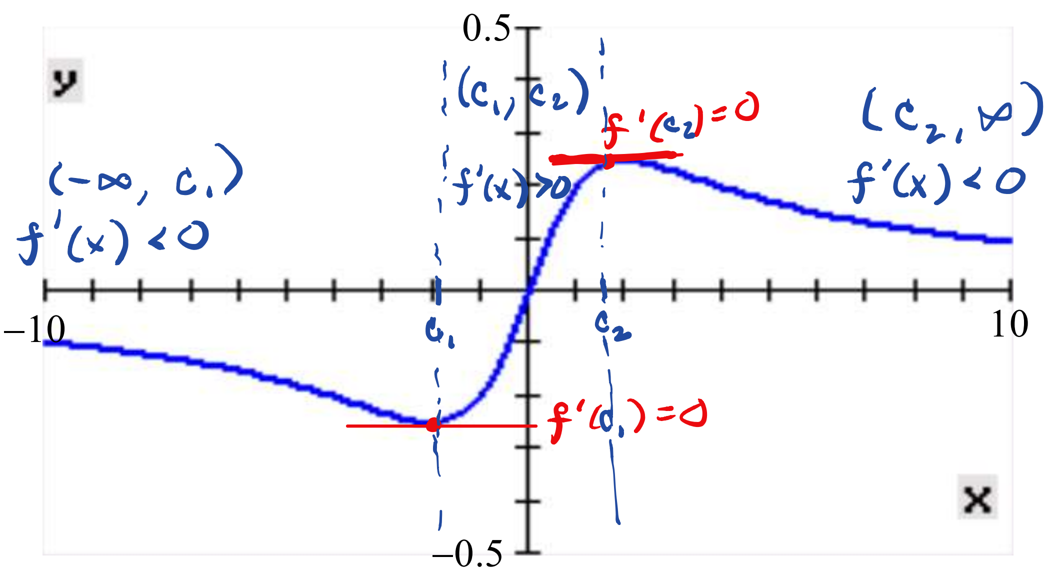
A function f is **increasing** on an interval if for any two numbers x_1 and x_2 in the in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function f is **decreasing** on an interval if for any two numbers x_1 and x_2 in the in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

THEOREM: TEST FOR INCREASING AND DECREASING FUNCTION INTERVALS

Let f be a function that is continuous on the closed interval $[a,b]$, and differentiable on the open interval (a,b).

1. If $f'(x) > 0$ for all x in (a,b), then f is **increasing** on (a,b).
2. If $f'(x) < 0$ for all x in (a,b), then f is **decreasing** on (a,b).
3. If $f'(x) = 0$ for all x in (a,b), then f is **constant** on (a,b).



GUIDELINES FOR FINDING INTERVALS ON WHICH A FUNCTION IS INCREASING OR DECREASING

Let f be continuous on the (a,b). To find the open intervals on which f is increasing or decreasing, use the following steps.

1. Locate the critical numbers of f in (a,b), and use these numbers to determine test intervals.
2. Determine the sign of $f'(x)$ at one test value in each of the intervals.
3. Use the test for increasing and decreasing functions to determine whether f is increasing or decreasing on each interval.

These guidelines are also valid if the interval (a,b) is replaced by an interval of the form $(-\infty,b)$, (a,∞), or $(-\infty,\infty)$.

Example 1: Identify the open intervals on which the function is increasing or decreasing.

$f(x) = 27x - x^3$

a. Find the critical numbers of f. f is diff. on $(-\infty, \infty)$

$f'(x) = 27 - 3x^2$

$0 = 27 - 3x^2$

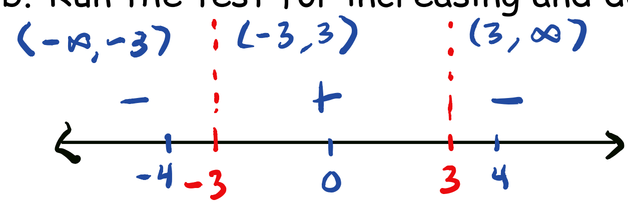
$3x^2 = 27$

$x^2 = 9$

$x = \pm 3$

$c_1 = -3, \; c_2 = 3$

b. Run the test for increasing and decreasing function intervals.



$f'(x) = 27 - 3x^2$

$f'(4) = 27 - 48 < 0$

$f'(-4) = 27 - 48 < 0$

$f'(0) = 27 - 0 > 0$

i. Find the open interval(s) on which the function is decreasing.

$(-\infty, -3) \cup (3, \infty)$

ii. Find the open interval(s) on which the function is increasing.

$(-3, 3)$

Example 2: Identify the open intervals on which the function is increasing or decreasing.

$f(x) = \cos^2 x - \cos x,\ 0 < x < 2\pi$

a. Find the critical numbers of f.

$\frac{d}{dx} f(x) = \frac{d}{dx}(\cos x)^2 - \frac{d}{dx}\cos x$

$f'(x) = 2(\cos x)'(-\sin x) - (-\sin x)$

$f'(x) = -2\sin x\cos x + \sin x$

$0 = \sin x(-2\cos x + 1)$

$\sin x = 0$ or $-2\cos x + 1 = 0$

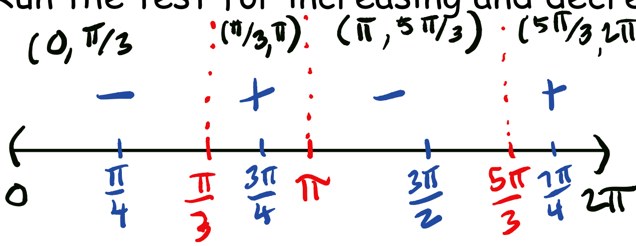
$\cos x = \frac{1}{2}$

$x = \pi$

$x = \pi/3,\ 5\pi/3$

$$\boxed{\begin{aligned} c_1 &= \pi/3 \\ c_2 &= \pi \\ c_3 &= 5\pi/3 \end{aligned}}$$

b. Run the test for increasing and decreasing intervals.



$f'(x) = -2\sin x\cos x + \sin x$

$f'(\pi/4) = -\cancel{2}\left(\frac{\sqrt{2}}{\cancel{2}}\right)\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} < 0$

$-1 \quad .7$

$f'(3\pi/4) = -2\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} > 0$

$f'(3\pi/2) = -2(-1)(0) + (-1) < 0$

$f'(7\pi/4) = -2\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2} > 0$

i. Find the open interval(s) on which the function is decreasing.

$\boxed{(0, \pi/3) \cup (\pi, \frac{5\pi}{3})}$

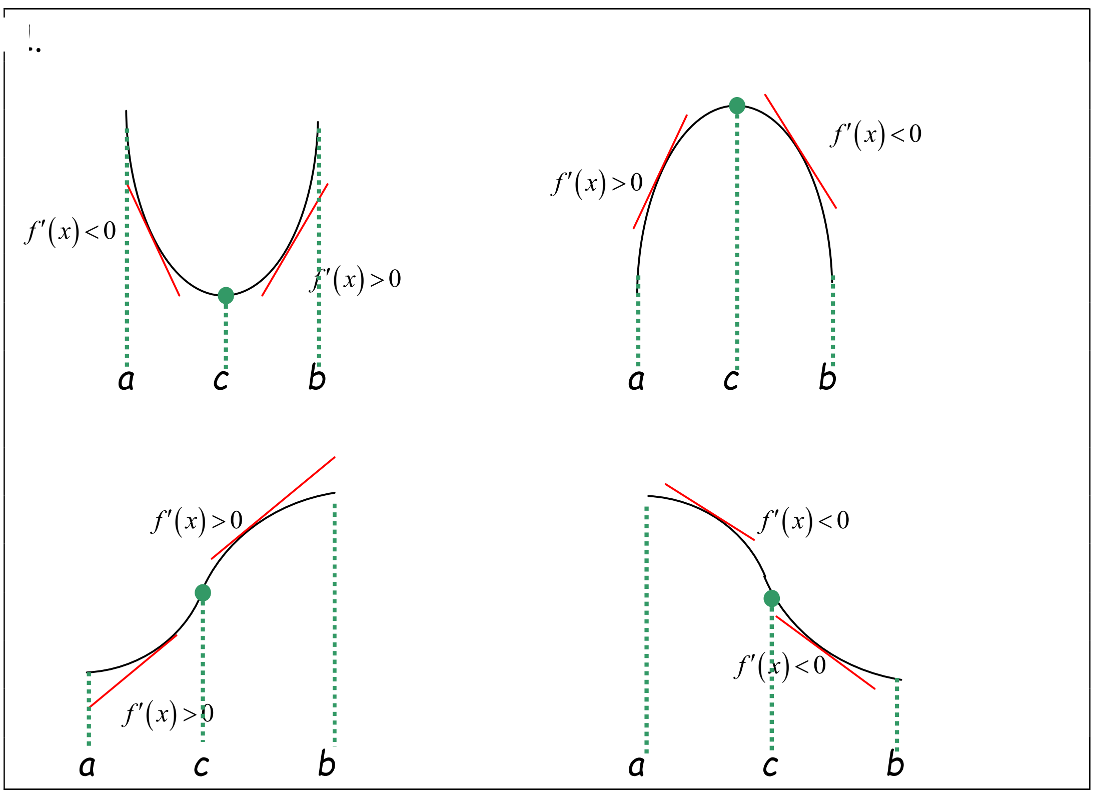
ii. Find the open interval(s) on which the function is increasing.

$\boxed{\left(\frac{\pi}{3}, \pi\right) \cup \left(\frac{5\pi}{3}, 2\pi\right)}$

THEOREM: THE FIRST DERIVATIVE TEST

Let c be a critical number of a function f that is continuous on an open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c, then f has a **relative minimum** at $(c, f(c))$.
2. If $f'(x)$ changes from positive to negative at c, then f has a **relative maximum** at $(c, f(c))$.
3. If $f'(x)$ is positive on both sides of c or negative on both sides of c, then $f(c)$ is neither a relative minimum or relative maximum.



Example 3: Consider the function $g(x) = x^{2/3} - 4$.

a. Find the critical numbers of g.

At $x=0$, there's a cusp, so g is not differentiable at $x=0$. $\boxed{c=0}$

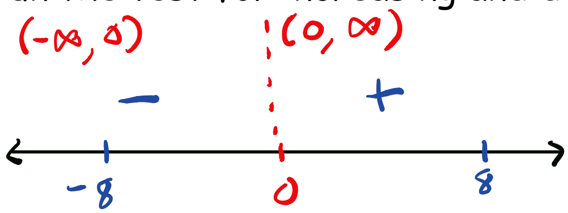
$g'(x) = \frac{2}{3}x^{-1/3}$

$0 = \dfrac{2}{3x^{1/3}}$

$\underbrace{0 = 2}_{}$

contradiction so no c.n. from zeros of derivative

b. Run the test for increasing and decreasing intervals.



$g'(x) = \dfrac{2}{3\sqrt[3]{x}}$

$g'(-8) = \dfrac{2}{3(-2)} < 0$

$g'(8) = \dfrac{2}{3(2)} > 0$

i. Find the open interval(s) on which the function is decreasing.

$\boxed{(-\infty, 0)}$

ii. Find the open interval(s) on which the function is increasing.

$\boxed{(0, \infty)}$

c. Apply the First Derivative Test. $g(x) = x^{2/3} - 4 \rightarrow g(0) = -4$

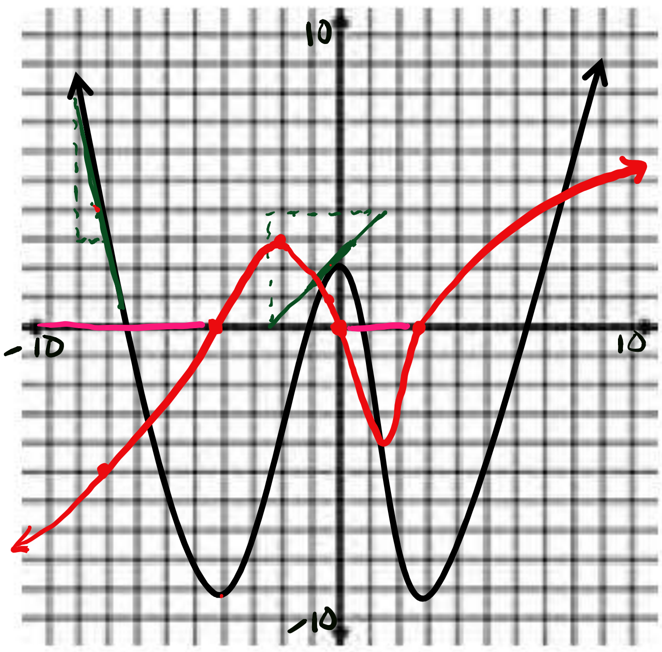
i. Identify all relative minima.

$\boxed{\text{relative minimum of } -4\text{; located at } (0,-4)}$

ii. Identify all relative maxima.

$\boxed{\text{NONE}}$

Example 4: The graph of a function f is given. The scale of each axis is from -10 to 10. Sketch a graph of the derivative of f .

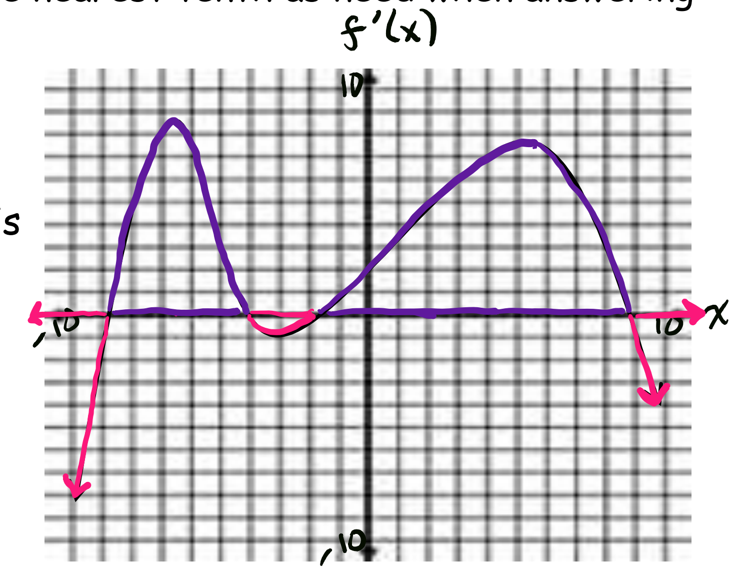


3.4: Concavity and the Second Derivative Test

When you are done with your homework you should be able to…

- π Determine intervals on which a function is concave upward or concave downward
- π Find any points of inflection of the graph of a function
- π Apply the Second Derivative Test to find relative extrema of a function

Warm-up: Consider the graph of f' shown below. The vertical and horizontal axes have a scale of -10 to 10. Round to the nearest tenth as need when answering the questions below.



a. Identify the interval(s) on which f is

i. increasing

$(-9, -4) \cup (-2, 9)$

ii. decreasing

$(-\infty, -9) \cup (-4, -2) \cup (9, \infty)$

b. Estimate the value(s) of x at which f has a relative

i. minimum

$x = -9, -2$

ii. maximum

$x = -4, 9$

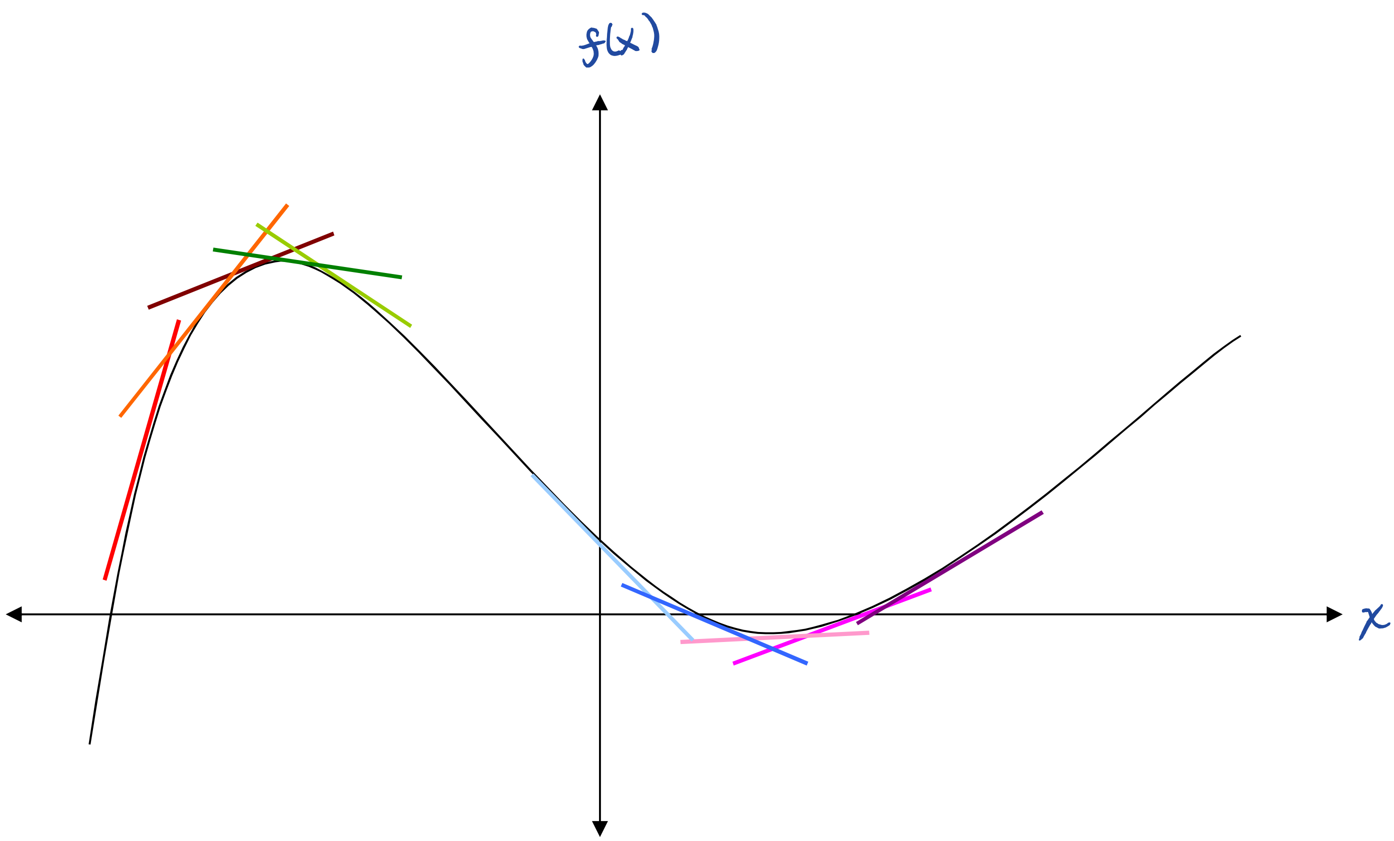
DEFINITION OF CONCAVITY

Let f be differentiable on an open interval I. The graph of f is **concave upward** on I if f' is increasing on the interval and **concave downward** on I if f' is decreasing on the interval.

THEOREM: TEST FOR CONCAVITY

Let f be a function whose second derivative exists on an open interval I.

1. If $f''(x) > 0$ for all x in I, then f is **concave upward** on I.
2. If $f''(x) < 0$ for all x in I, then f is **concave downward** on I.



Example 1: Identify the open intervals on which the function is concave upward or concave downward.

a. $y = -x^3 + 3x^2 - 2$ $\qquad y(1) = -1 + 3 - 2 = 0$

1. Find the <u>zeros</u> of the <u>second</u> derivative.

$y'(x) = -3x^2 + 6x$

$y''(x) = -6x + 6$

$0 = -6x + 6$

$6x = 6$

$x = 1$

2. Run the <u>test for concavity</u>.

$(-\infty, 1)$ $\quad$ $(1, \infty)$

$+$ $\quad$ $-$

0 $\quad$ 1 $\quad$ 2

$f''(x) = -6x + 6$

$f''(0) = 6 > 0$

$f''(2) = -6 < 0$

3. Conclusion.

y is concave upward on $(-\infty, 1)$ and concave downward on $(1, \infty)$.

Point of inflection at $(1, y(1)) = (1, 0)$

b. $f(x) = x + \dfrac{2}{\sin x}$, $(-\pi, \pi)$

1. Find the zeros of the second derivative.

$f(x) = x + 2\csc x$

$f'(x) = 1 - 2\csc x \cot x$

$f''(x) = -2[-\csc x \cot x \cot x + \csc x(-\csc^2 x)]$

$f''(x) = 2(\csc x \cot^2 x + \csc^3 x)$

$0 = 2\csc x(\cot^2 x + \csc^2 x)$

$2\csc x = 0$ or $\cot^2 x + \csc^2 x = 0$

$\cancel{\csc x = 0}$ extraneous

$\left(\dfrac{\cos x}{\sin x}\right)^2 + \left(\dfrac{1}{\sin x}\right)^2 = 0$

$\dfrac{\cos^2 x + 1}{\sin^2 x} = 0$

$\cos^2 x = -1$

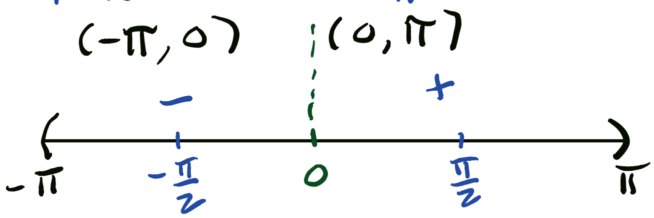
$\cos x = \pm i$

imaginary

$\boxed{\text{no real zeros}}$

2. Run the test for concavity.

There's a VA at $x = 0$.



$f''(x) = 2\csc x(\underbrace{\cot^2 x + \csc^2 x}_{\text{always } +})$

$f''(-\tfrac{\pi}{2}) = 2\csc(-\pi/2)(\text{pos. \#})$
$= 2(-\csc \pi/2)(\text{pos. \#})$
$= -2(1) < 0$

$f''(\pi/2) = 2(\csc \pi/2)(\text{pos \#})$
$= 2(1)(\text{pos. \#}) > 0$

3. Conclusion.

f is concave downward on $(-\pi, 0)$ and concave upward on $(\pi, 0)$.

DEFINITION OF POINT OF INFLECTION

Let f be a function that is continuous on an open interval and let c be an element in the interval. If the graph of f has a tangent line at the point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward or from downward to upward at the point.

THEOREM: POINTS OF INFLECTION

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or f'' does not exist at $x = c$. $c \in$ Domain of f.

Example 2: Consider the function $g(x) = 2x^4 - 8x + 3$.

a. Discuss the concavity of the graph of g.

1. Find the zeros of the second derivative.

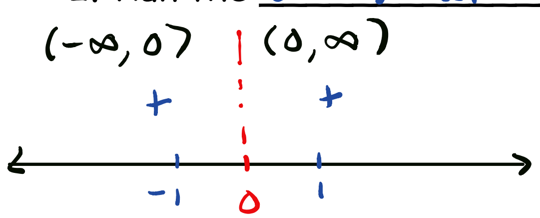
$g'(x) = 8x^3 - 8$

$g''(x) = 24x^2$

$0 = 24x^2$

$0 = x$

2. Run the test for concavity.



$g''(x) = 24x^2$

$g''(-1) = 24(1) > 0$

$g''(1) = 24(1) > 0$

3. Conclusion.

g is concave upward on $(-\infty, 0) \cup (0, \infty)$. g is never concave downward.

b. Find all points of inflection.

$\boxed{\text{NONE}}$

* There's no change in concavity at $x = 0$.

THEOREM: SECOND DERIVATIVE TEST

Let f be a function such that $f'(c)=0$ and the second derivative of f exists on an open interval containing c.

1. If $f''(c)>0$, then f has a **relative minimum** at $(c, f(c))$.
2. If $f''(c)<0$, then f has a **relative maximum** at $(c, f(c))$.
3. If $f''(c)=0$, the test FAILS and you need to run the FIRST DERIVATIVE TEST.

Example 3: Find all relative extrema. Use the Second Derivative Test where applicable.

a. $f(x)=x^3-5x^2+7x$

Find C.N. for f

$f'(x) = 3x^2-10x+7$

$0 = 3x^2-10x+7$

$0 = 3x^2-7x-3x+7$

$0 = x(3x-7)-1(3x-7)$

$0 = (3x-7)(x-1)$

$0 = 3x-7$ or $0 = x-1$

$x = \frac{7}{3}$ or $x = 1$

$c_1 = 1, \ c_2 = \frac{7}{3}$

21, −7, −3, −10

$f(1) = (1)^3-5(1)^2+7(1) = 3$

$f(7/3) = (7/3)^3-5(7/3)^2+7(7/3)$

$= 343/27 - 5(49/9) + 49/3$

$= \frac{49}{27}$

Find $f''(x)$ and evaluate at c_1, c_2

$f'(x) = 3x^2-10x+7$

$f''(x) = 6x-10$

$f''(1) = 6(1)-10 = -4 < 0$

$\boxed{\text{so there's a relative max at } (1, f(1)) = (1,3).}$

$f''(7/3) = \cancel{6}^{2}(\frac{7}{\cancel{3}})-10 = 4 > 0$

$\boxed{\text{so there's a relative minimum at } (7/3, f(7/3)) = (7/3, \frac{49}{27})}$

b. $f(x)=\frac{x}{x-1}$

$$f'(x)=\frac{1(x-1)-x(1)}{(x-1)^2}$$

$$f'(x)=\frac{x-1-x}{(x-1)^2}$$

$$f'(x)=\frac{-1}{(x-1)^2} \rightarrow 0=-1$$ contradiction, so no critical #'s.

$$f'(x)=-(x-1)^{-2}$$

There's no critical numbers, so there can't be any relative extrema.

Example 4: Sketch the graph of a function f having the given characteristics.

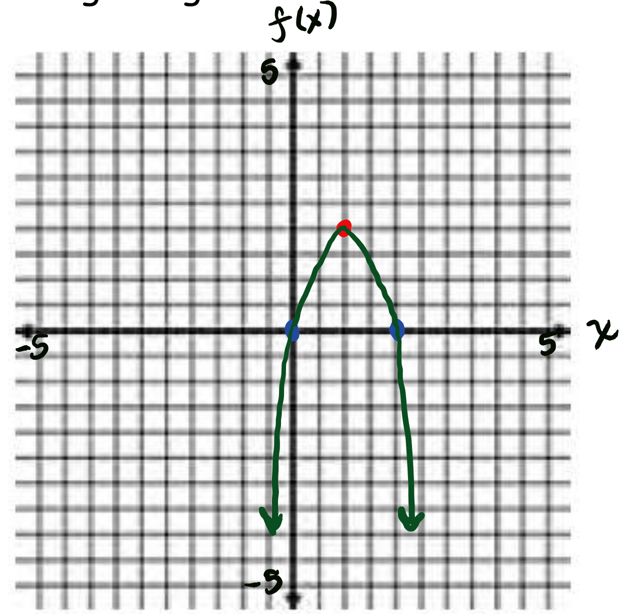
$f(0) = f(2) = 0$ → (0,0) and (2,0) are points on f

$f'(x) > 0$ if $x < 1$ → f is ↑ on $(-\infty, 1)$

$f'(1) = 0$ → rel min or max at x = 1

$f'(x) < 0$ if $x > 1$ → f is ↓ on $(1, \infty)$

$f''(x) < 0$ concave downward on $(-\infty, \infty)$



3.5: Limits at Infinity

When you finish your homework you should be able to…

- π Determine finite limits at infinity
- π Determine the horizontal asymptotes, if any, of the graph of a function
- π Determine infinite limits at infinity

Warm-up: Evaluate the following limits analytically

a. $\lim_{x \to 1^+} \frac{3}{1-x} = \boxed{-\infty}$ DNE

$\frac{3}{1-x}$ at 1.01

$\frac{3}{1-1.01} = \frac{3}{-\frac{1}{100}} = -300$

b. $\lim_{t \to 0} \frac{\sin 3t}{t} \cdot \frac{3}{3}$

$= 3 \lim_{t \to 0} \frac{\sin 3t}{3t}$

$= 3(1)$

$= \boxed{3}$

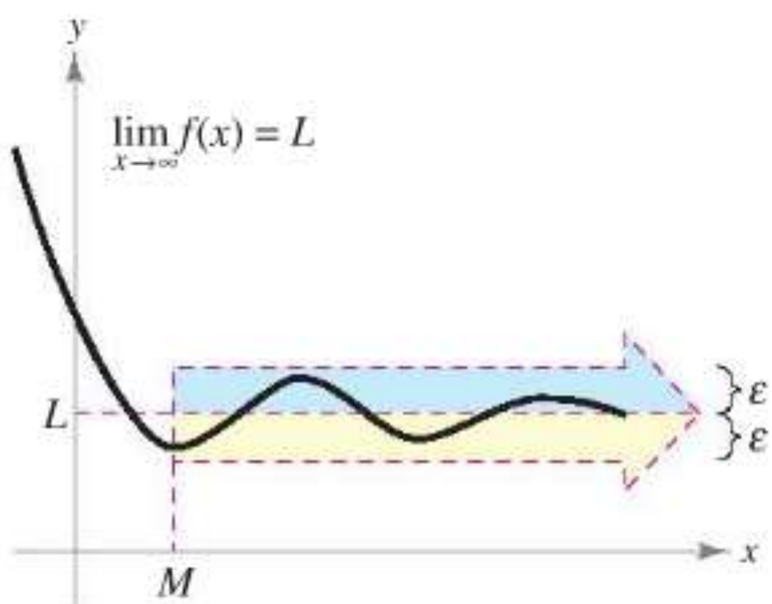
LIMITS AT INFINITY

This section discusses the ___end___ behavior of a function on an ___infinite___ interval.

DEFINITION OF LIMITS AT INFINITY

Let L be a real number.

1. The statement $\lim\limits_{x\to\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $M > 0$, such that $|f(x) - L| < \varepsilon$ whenever $x > M$.
2. The statement $\lim\limits_{x\to-\infty} f(x) = L$ means that for each $\varepsilon > 0$ there exists an $N < 0$, such that $|f(x) - L| < \varepsilon$ whenever $x < N$.



DEFINITION OF A HORIZONTAL ASYMPTOTE

The line $y = L$ is a **horizontal asymptote** of the graph of f if $\lim\limits_{x\to-\infty} f(x) = L$ or $\lim\limits_{x\to\infty} f(x) = L$.

THEOREM: LIMITS AT INFINITY

If r is a positive rational number, then

$$\lim_{x\to\infty} \frac{c}{x^r} = 0.$$

Furthermore, if x^r is defined when $x<0$, then

$$\lim_{x\to-\infty} \frac{c}{x^r} = 0.$$

DEFINITION OF INFINITE LIMITS AT INFINITY

Let f be a function defined on the interval (a,∞).

1. The statement $\lim_{x\to\infty} f(x) = \infty$ means that for each $M>0$ there is a corresponding number $N>0$, such that $f(x)>M$ whenever $x>N$.
2. The statement $\lim_{x\to\infty} f(x) = -\infty$ means that for each $M<0$ there is a corresponding number $N>0$, such that $f(x)<M$ whenever $x>N$.

GUIDELINES FOR FINDING LIMITS AT +/- INFINITY

1. If the degree of the numerator is **less than** the degree of the denominator, then the limit of the rational function is 0.
2. If the degree of the numerator is **equal to** the degree of the denominator, then the limit of the rational function is the ratio of the leading coefficients.
3. If the degree of the numerator is **greater than** the degree of the denominator, then the limit of the rational function is plus or minus infinity, hence it does not exist.

Example 1: Find the limit.

a. $\lim\limits_{x\to-\infty}\left(\frac{5}{x}-\frac{x}{3}\right) = \lim\limits_{x\to-\infty}\frac{5}{x^1} - \lim\limits_{x\to-\infty}\frac{x}{3}$

$= 0 - (-\infty)$

$= \infty$ DNE

limits at infinity theorem
$c = 5$
$r = 1$

b. $\lim\limits_{x\to\infty}\left(\frac{x^2+3}{2x^2-1}\right) = \dfrac{\lim\limits_{x\to\infty}\frac{x^2+3}{x^2}}{\lim\limits_{x\to\infty}\frac{2x^2-1}{x^2}}$

$c=3, r=2$

$= \dfrac{\lim\limits_{x\to\infty} 1 + \lim\limits_{x\to\infty}\frac{3}{x^2}}{\lim\limits_{x\to\infty} 2 - \lim\limits_{x\to\infty}\frac{1}{x^2}}$

limits at infinity theorem

$c=1, r=2$

$= \dfrac{1+0}{2-0}$

$= \boxed{\frac{1}{2}}$

so the line $y=\frac{1}{2}$ is a horizontal asymptote.

c. $\lim_{x\to\infty}\sqrt{\frac{x^4-1}{x^3-1}} = \lim_{x\to\infty} \frac{\sqrt{x^4-1}}{\sqrt{x^3-1}}$

$$= \frac{\sqrt{\lim_{x\to\infty} \frac{x^4-1}{x^3}}}{\sqrt{\lim_{x\to\infty} \frac{x^3-1}{x^3}}}$$

$$= \frac{\sqrt{\lim_{x\to\infty} x - \lim_{x\to\infty} \frac{1}{x^3}}}{\sqrt{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x^3}}}$$

$$= \frac{\sqrt{\infty - 0}}{\sqrt{1-0}}$$

$$= \sqrt{\infty}$$

$$= \infty \quad \text{DNE}$$

d. $\lim_{x\to\infty}\cos\frac{1}{x} = \cos \lim_{x\to\infty} \frac{1}{x}$

$$= \cos 0$$

$$= \boxed{1}$$

There's a HA at $y = 1$.

$\sqrt{x^2} = \pm x$

e. $\lim_{x\to\infty} \frac{(-3x+1)/\sqrt{x^2}}{\sqrt{(x^2+x)/x^2}} = \frac{\lim_{x\to\infty} \pm 3 + \lim_{x\to\infty} \frac{1}{\pm x}}{\lim_{x\to\infty} \sqrt{1+1/x}}$

$= \frac{\pm 3 + 0}{\sqrt{\lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{1}{x}}}$

$= \frac{\pm 3}{\sqrt{1+0}}$

$= \frac{\pm 3}{1}$

$= \boxed{\pm 3}$

There's a HA at $y=-3$ and $y=3$.

f. $\lim_{x\to\infty} \frac{x/x^2}{(x^2-1)/x^2} = \frac{\lim_{x\to\infty} \frac{1}{x}}{\lim_{x\to\infty} 1 - \lim_{x\to\infty} \frac{1}{x^2}}$

$= \frac{0}{1-0}$

$= \boxed{0}$

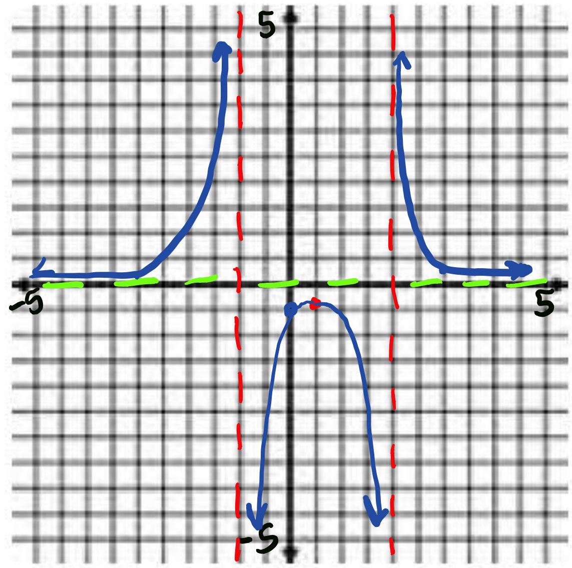
There's a HA at $y=0$.

⑦ Find extra points

$f(3) = \frac{1}{9-3-2} = \frac{1}{4}$, $f(-3) = \frac{1}{9+3-2} = \frac{1}{10}$

Example 2: Sketch the graph of the equation using extrema, intercepts, symmetry, and asymptotes. Then use a graphing utility to verify your result.

a. $f(x) = \frac{1}{x^2 - x - 2}$



④ Find the C.N.'s

$f(x) = (x^2-x-2)^{-1}$

$f'(x) = -1(x^2-x-2)^{-2}(2x-1)$ chain rule

$f'(x) = -\frac{2x-1}{(x^2-x-2)^2}$

$0 = 2x-1$

$x = \frac{1}{2}$

$c = \frac{1}{2}$

② Find VA's

$x^2-x-2=0$

$(x-2)(x+1)=0$

$x-2=0$ or $x+1=0$

$x=2$ or $x=-1$

$\boxed{x=-1 \quad x=2}$

⑤ Test for ↑, ↓ intervals and find relative extrema

$(-\infty,-1)$; $(-1,\frac{1}{2})$; $(\frac{1}{2},2)$; $(2,\infty)$

$+$; $+$; $-$; $-$

$-2 \quad -1 \quad 0 \quad \frac{1}{2} \quad 1 \quad 2 \quad 3$

$f'(x) = -\frac{2x-1}{(x^2-x-2)^2}$

$f'(-2) = -\frac{-5}{\text{pos.\#}} > 0$

$f'(0) = -\frac{-1}{\text{pos \#}} > 0$

$f'(1) = -\frac{1}{\text{pos\#}} < 0$

$f'(3) = -\frac{5}{\text{pos\#}} < 0$

f is ↓ on $(\frac{1}{2},2) \cup (2,\infty)$

f is ↑ on $(-\infty,-1) \cup (-1,\frac{1}{2})$

$f(\frac{1}{2}) = 1/[(1/2)^2 - 1/2 - 2] = -4/9$

rel. max at $(\frac{1}{2}, -\frac{4}{9})$

③ Find HA's

$\lim\limits_{x\to\infty} \frac{1/x^2}{\frac{x^2-x-2}{x^2}} = \frac{\lim\limits_{x\to\infty} \frac{1}{x^2}}{\lim\limits_{x\to\infty} (1-\frac{1}{x}-\frac{2}{x^2})}$

$= \frac{0}{1-0-0}$

$= 0$

$\boxed{y=0}$

⑥ concavity

$f'(x) = -\frac{2x-1}{(x^2-x-2)^2}$

$f''(x) = -\left[\frac{2(x^2-x-2)^{\cancel{2}\,1} - (2x-1)2(x^2-x-2)^{1}(2x-1)}{(x^2-x-2)^{\cancel{4}\,3}}\right]$

$f''(x) = -\frac{2x^2-2x-4-2(4x^2-4x+1)}{(x^2-x-2)^3}$

$f''(x) = -\frac{-6x^2+6x-6}{(x^2-x-2)^3}$

$f''(x) = \frac{6(x^2-x+1)}{(x^2-x-2)^3}$

Discriminant of quad formula

$b^2-4ac \to (-1)^2-4=-3<0$

no real zeros

① Find intercepts

x-int: $0 = \frac{1}{x^2-x-2}$

$0=1$ false

$\boxed{\text{NONE}}$

y-int: $f(0) = \frac{1}{0^2-0-2} = -\frac{1}{2}$

$\boxed{(0,-1/2)}$

$f''(-2) = \frac{6[(-2)^2-2+1]}{[(-2)^2+2-2]^3} > 0$

$f''(0) = \frac{6}{-8} < 0$

$f''(3) = \frac{6(9-3+1)}{(9-3-2)^3} > 0$

$(-\infty,-1)$; $(-1,2)$; $(2,\infty)$

$+$; $-$; $+$

$-2 \quad -1 \quad 0 \quad 2 \quad 3$

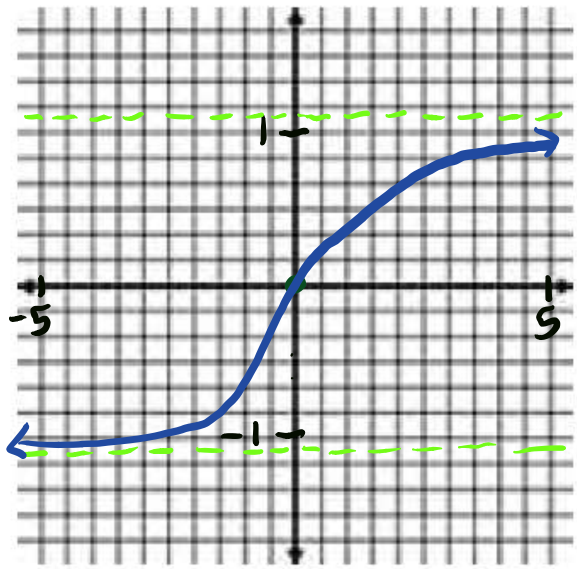
$\boxed{f \text{ is concave upward on } (-\infty,-1) \cup (2,\infty)}$

f is concave downward on (-1, 2)
no points of inflection

Domain: $3x^2+1>0$
$(-\infty,\infty)$

b. $$h(x)=\frac{2x}{\sqrt{3x^2+1}}$$

$f(x)=2x$
$g(x)=\sqrt{3x^2+1}$
for quotient rule



① Intercepts

x-int: $0=2x \rightarrow x=0$
$\boxed{(0,0)}$

y-int: $f(0)=\frac{2\cdot 0}{\sqrt{3\cdot 0^2+1}}=0$
$\boxed{(0,0)}$

② Asymptotes

VA: $\sqrt{3x^2+1}=0$
$3x^2+1=0$
$x^2=-\frac{1}{3}$ imaginary
$\boxed{\text{NONE}}$

HA: $$\lim_{x\to\infty}\frac{2x/|x|}{\sqrt{\frac{3x^2+1}{x^2}}}=\frac{\lim\limits_{x\to\pm\infty}2(\pm 1)}{\sqrt{\lim\limits_{x\to\infty}\left(3+\frac{1}{x^2}\right)}}=\frac{\pm 2}{\sqrt{3}}\approx \pm 1.15$$

$\boxed{\sqrt{x^2}=|x|}$

$$\boxed{\begin{aligned}y&=-\tfrac{2}{\sqrt{3}}\approx -1.15\\ y&=\tfrac{2}{\sqrt{3}}\approx 1.15\end{aligned}}$$

③ Find C.N.

$$h'(x)=\frac{2(3x^2+1)^{1/2}-2x\left[\frac{1}{2}(3x^2+1)^{-1/2}\cdot 6x\right]}{3x^2+1}$$

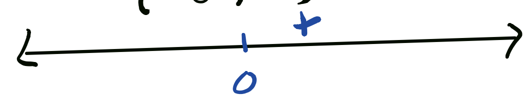
$$h'(x)=\frac{2(3x^2+1)^{-1/2}\left[(\cancel{3x^2}+1)^1-\cancel{3x^2}\right]}{3x^2+1}$$

$$h'(x)=\frac{2}{(3x^2+1)^{3/2}}$$

$0=2$ false so no zeros
and everywhere differentiable
so no C.N.'s $\boxed{\text{NONE}}$

④ Test for ↑, ↓ intervals

$(-\infty,\infty)$



$$h'(0)=\frac{2}{(0+1)^{3/2}}=2$$

no relative extrema
h is increasing on $(-\infty,\infty)$

⑤ concavity

$$h'(x)=\frac{2}{(3x^2+1)^{3/2}}$$

$h'(x)=2(3x^2+1)^{-3/2}$

$h''(x)=2\left[-\frac{3}{2}(3x^2+1)^{-5/2}\cdot 6x\right]$

$$h''(x)=\frac{-18x}{(3x^2+1)^{5/2}}$$

$0=-18x \rightarrow x=0$
$(-\infty,0)$; $(0,\infty)$

$h''(-1)=\frac{18}{\text{pos \#}}>0$ h is concave up on $(-\infty,0)$

$h''(1)=\frac{-18}{\text{pos \#}}<0$ h is concave down on $(0,\infty)$

POI: $(0,h(0))=\boxed{(0,0)}$

3.6: A Summary of Curve Sketching

When you finish your homework you should be able to…

π Curve sketch using methods from Calculus

Example 1: Sketch the graph of the equation by hand. If a particular characteristic of the graph does not occur, write "none".

$$f(x)=\frac{x}{x^2+1}$$

a. Intercepts (write as ordered pairs)

x-int: $0 = x/(x^2+1) \to 0 = x$

y-int: $f(0) = \frac{0}{0^2+1} = 0$

i. x-intercept: $\boxed{(0,0)}$

ii. y-intercept: $\boxed{(0,0)}$

b. Vertical Asymptote(s)

$x^2+1=0 \to x^2=-1$ imaginary

$\boxed{\text{NONE}}$

c. Behavior at vertical asymptote(s)

$\boxed{\text{NONE}}$

d. Horizontal Asymptote(s)

$$\lim_{x\to\infty}\frac{x/x^2}{x^2+1 \,/\, x^2} = \lim_{x\to\infty}\frac{1/x}{1+1/x^2} = \frac{0}{1+0} = 0$$

$\boxed{y=0}$

e. Run the test for increasing/decreasing intervals

$$f'(x) = \frac{1(x^2+1) - x(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$(-\infty, -1)$ | $(-1, 1)$ | $(1, \infty)$

$-$ | $+$ | $-$

-2 $\quad -1 \quad 0 \quad 1 \quad 2$

$$0 = \frac{1-x^2}{(x^2+1)^2}$$

$$0 = 1 - x^2$$

$$x^2 = 1$$

$$x = \pm 1$$

$$c = \pm 1$$

$$f'(-2) = \frac{1-(-2)^2}{\text{pos \#}} = \frac{-3}{\text{pos\#}} < 0$$

$$f'(0) = \frac{1-(0)^2}{\text{pos\#}} = \frac{1}{\text{pos\#}} > 0$$

$$f'(2) = f'(-2) < 0$$

i. f is increasing on $\boxed{(-1, 1)}$

ii. f is decreasing on $\boxed{(-\infty, -1) \cup (1, \infty)}$

f. Find the ordered pairs where relative extrema occur.

$$f(-1) = \frac{(-1)}{(-1)^2+1} = -\frac{1}{2}$$

$$f(1) = \frac{(1)}{(1)^2+1} = \frac{1}{2}$$

i. Relative minima: $\boxed{(-1, -\tfrac{1}{2})}$

ii. Relative maxima: $\boxed{(1, \tfrac{1}{2})}$

$\sqrt{3} \approx 1.7$

$(-\infty, -\sqrt{3}) \;|\; (-\sqrt{3}, 0) \;|\; (0, \sqrt{3}) \;|\; (\sqrt{3}, \infty)$

$-$ $\quad +$ $\quad -$ $\quad +$

$-2 \quad -\sqrt{3} \quad -1 \quad 0 \quad 1 \quad \sqrt{3} \quad 2$

g. Test for concavity.

$$f'(x) = \frac{1-x^2}{(x^2+1)^2}$$

$$f''(x) = \frac{-2x(x^2+1)^{\cancel{2}\,1} - (1-x^2)\left[2\cancel{(x^2+1)}'(2x)\right]}{(x^2+1)^{\cancel{4}\,3}}$$

$4x$

$$f''(x) = \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3}$$

$$f''(x) = \frac{2x^3 - 6x}{(x^2+1)^3}$$

$0 = 2x^3 - 6x \rightarrow 0 = 2x(x^2-3) \rightarrow 2x = 0$ or $x^2 - 3 = 0 \rightarrow x = 0$ or $x = \pm\sqrt{3}$

$$f''(-2) = \frac{2(-2)^3 - 6(-2)}{\text{pos\#}} < 0$$

$$f''(-1) = \frac{2(-1)^3 - 6(-1)}{\text{pos\#}} > 0$$

$$f''(1) = \frac{2(1)^3 - 6(1)}{\text{pos\#}} < 0$$

$$f''(2) = \frac{2(2)^3 - 6(2)}{\text{pos\#}} > 0$$

i. f is concave upward on $\boxed{(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)}$

ii. f is concave downward on $\boxed{(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})}$

h. Find the points of inflection.

$$f(0) = \frac{0}{0^2+1} = 0$$

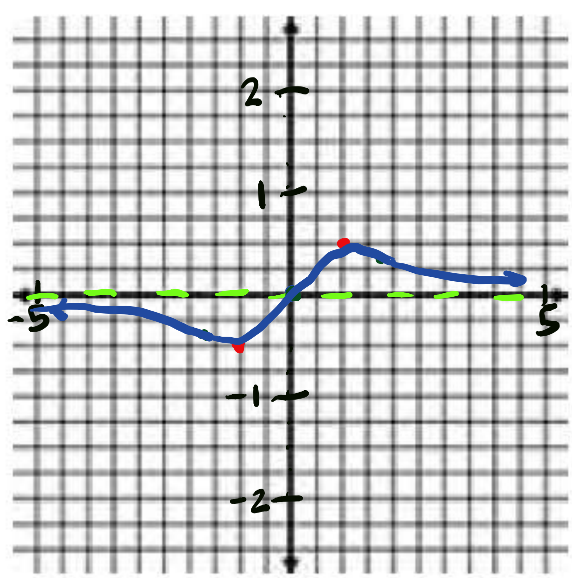
$$f(\sqrt{3}) = \frac{\sqrt{3}}{(\sqrt{3})^2+1} = \frac{\sqrt{3}}{4}$$

$$f(-\sqrt{3}) = \frac{-\sqrt{3}}{(-\sqrt{3})^2+1} = -\frac{\sqrt{3}}{4}$$

$$\boxed{\begin{array}{l}(0,0)\\ (-\sqrt{3}, -\frac{\sqrt{3}}{4})\\ (\sqrt{3}, \sqrt{3}/4)\end{array}}$$

$\frac{\sqrt{3}}{4} \approx 0.43$

i. Sketch the graph by hand.



Example 2: Sketch the graph of the equation by hand. If a particular characteristic of the graph does not occur, write "none".

$$f(x)=\frac{2x^2-5x+5}{x-2}$$

a. Intercepts (write as ordered pairs)

x-int: $0 = 2x^2-5x+5$

$x = \frac{5 \pm \sqrt{25-4(2)(5)}}{4}$

$x = \frac{5 \pm \sqrt{-15}}{4} \to$ imaginary

y-int: $f(0) = \frac{2(0)^2-5(0)+5}{(0)-2} = -\frac{5}{2}$

i. x-intercept: $\boxed{\text{NONE}}$

ii. y-intercept: $\boxed{\left(0,-\frac{5}{2}\right)}$

b. Vertical Asymptote(s)

$x-2=0 \to \boxed{x=2}$

c. Behavior at vertical asymptote(s)

Skip

d. Horizontal Asymptote(s)

$$\lim_{x\to\infty} \frac{(2x^2-5x+5)/x}{\frac{x-2}{x}} = \lim_{x\to\infty} \frac{2x-5+5/x}{1-2/x}$$

$$= \frac{\infty-5+0}{1-0}$$

$$= \infty$$

$\boxed{\text{NONE}}$

e. Run the test for increasing/decreasing intervals

$$f(x)=\frac{2x^2-5x+5}{x-2}$$

$$f'(x)=\frac{(4x-5)(x-2)-(2x^2-5x+5)(1)}{(x-2)^2}$$

$$f'(x)=\frac{4x^2-13x+10-2x^2+5x-5}{(x-2)^2}$$

$$f'(x)=\frac{2x^2-8x+5}{(x-2)^2}$$

$$0=2x^2-8x+5$$

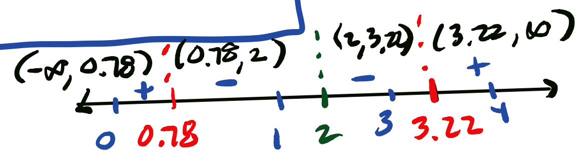
$$x=\frac{8\pm\sqrt{64-4(2)(5)}}{4}$$

$$x=\frac{8\pm\sqrt{24}}{4}$$

$$x=\frac{8\pm2\sqrt{6}}{4}$$

$$x=\frac{4\pm\sqrt{6}}{2}$$

$c_1=\frac{4-\sqrt{6}}{2}\approx0.78$, $c_2=\frac{4+\sqrt{6}}{2}\approx3.22$



$f'(0)=\frac{5}{pos\#}>0$

$f'(1)=\frac{-1}{pos\#}<0$

$f'(3)=\frac{-1}{pos\#}<0$

$f'(4)=\frac{5}{pos\#}>0$

i. f is increasing on $\boxed{(-\infty,0.78)\cup(3.22,\infty)}$

ii. f is decreasing on $\boxed{(0.78,2)\cup(2,3.22)}$

f. Find the ordered pairs where relative extrema occur.

$f(0.78)\approx1.90$

$f(3.22)\approx7.90$

i. Relative minima: $\boxed{(3.22,7.90)}$

ii. Relative maxima: $\boxed{(0.78,1.90)}$

$$f'(x) = \frac{2x^2 - 8x + 5}{(x-2)^2}$$

g. Test for concavity.

$$f''(x) = \frac{(4x-8)(x-2)^{\cancel{2}\,1} - (2x^2-8x+5)[2(\cancel{x-2})'(1)]}{(x-2)^{\cancel{4}\,3}}$$

$$f''(x) = \frac{\cancel{4x^2} - \cancel{16x} + 16 - \cancel{4x^2} + \cancel{16x} - 10}{(x-2)^3}$$

$$f''(x) = \frac{6}{(x-2)^3}$$

$0 = 6$ false

$(-\infty, 2)$ | $(2, \infty)$

$-$ | $+$

0, 2, 3

$$f''(0) = \frac{6}{-8} < 0$$

$$f''(3) = \frac{6}{1} > 0$$

i. f is concave upward on $(2, \infty)$

ii. f is concave downward on $(-\infty, 2)$

h. Find the points of inflection.

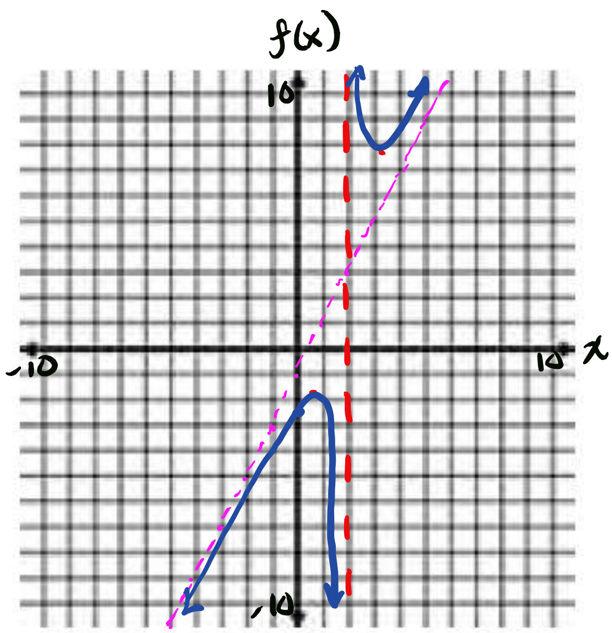
$\boxed{\text{NONE}}$ since at $x = 2$ there's a VA

i. Sketch the graph by hand.

Oblique asymptote

$$\begin{array}{r} 2x - 1 + \frac{3}{x-2} \\ (x-2)\overline{)\,2x^2 - 5x + 5} \\ -(2x^2 - 4x) \quad \downarrow \\ -x + 5 \\ -(-x + 2) \\ \hline 3 \end{array}$$

$y = 2x - 1$



Example 3: Sketch the graph of the equation by hand. If a particular characteristic of the graph does not occur, write "none".

$$f(x) = 3(x-1)^{2/3}$$

a. Intercepts (write as ordered pairs)

i. x-intercept: _______________

ii. y-intercept: _______________

b. Vertical Asymptote(s)

c. Behavior at vertical asymptote(s)

d. Horizontal Asymptote(s)

e. Run the test for increasing/decreasing intervals

i. f is increasing on ________________________________

ii. f is decreasing on ________________________________

f. Find the ordered pairs where relative extrema occur.

i. Relative minima: ____________________

ii. Relative maxima: ____________________

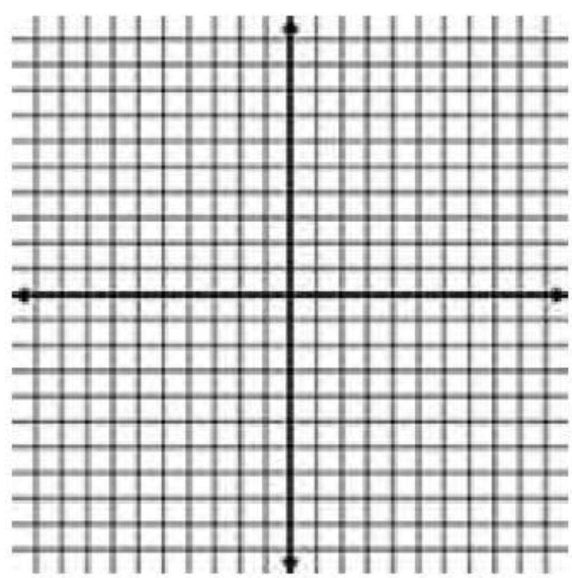
g. Test for concavity.

i. f is concave upward on ___________________________________

ii. f is concave downward on _________________________________

h. Find the points of inflection.

i. Sketch the graph by hand.



Example 4: Sketch the graph of the equation by hand. If a particular characteristic of the graph does not occur, write "none".

$$f(x) = -x + 2\cos x,\ [0, 2\pi]$$

a. Intercepts (write as ordered pairs)

i. x-intercept: _______________

ii. y-intercept: _______________

b. Vertical Asymptote(s)

c. Behavior at vertical asymptote(s)

d. Horizontal Asymptote(s)

e. Run the test for increasing/decreasing intervals

i. f is increasing on ________________________________

ii. f is decreasing on ________________________________

f. Find the ordered pairs where relative extrema occur.

i. Relative minima: _____________________

ii. Relative maxima: _____________________

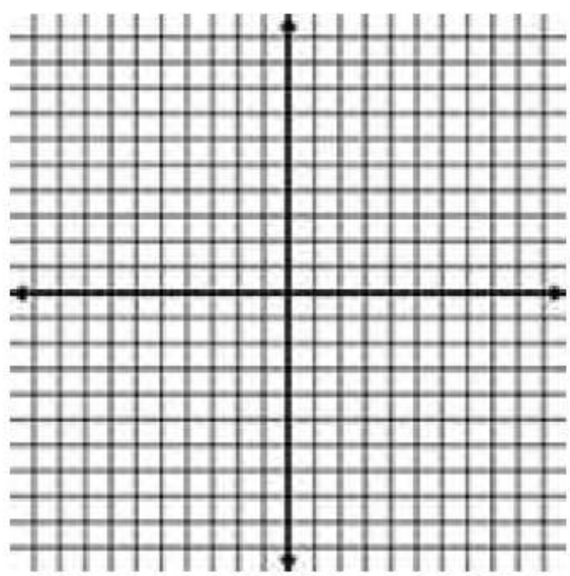
g. Test for concavity.

i. f is concave upward on ___________________________________

ii. f is concave downward on _________________________________

h. Find the points of inflection.

i. Sketch the graph by hand.



3.7: Optimization Examples

When you finish your homework you should be able to…

- π Solve applied minimum and maximum problems

Guidelines for Solving Applied Minimum and Maximum Problems

1. **Analyze:** Identify all given quantities and all quantities to be determined. MAKE A SKETCH!!!
2. Write a **primary equation** for the quantity that is to be maximized or minimized.
3. **Reduce the primary equation** to one having a single independent variable. You may need to use secondary equations relating the independent variables of the primary equation.
4. Determine the **feasible domain** of the primary equation.
5. **Optimize** (Find zeros of the critical numbers)
6. **Verify** (Use first or second derivative test)
7. Find all maximum or minimum values by back substitution.
8. State your **conclusion** in words (Does your conclusion make sense).

1. Find two positive numbers that satisfy the following requirements:
   The sum of the first number squared and the second is 27 and the product is a maximum.

Analyze: ______________________________

____

Primary Equation: ______________________ **Feasible Domain:**

Reduce to 1 variable: __________________

Optimize and Verify:

Conclusion:

2. A woman has two dogs that do not get along. She has a theory that if they are housed in kennels right next to each other, they'll get used to each other. She has 200 feet of fencing with which to enclose two adjacent rectangular kennels. What dimensions should be used so that the enclosed area will be a maximum?

Analyze:

Primary Equation: ____________________

Feasible Domain:

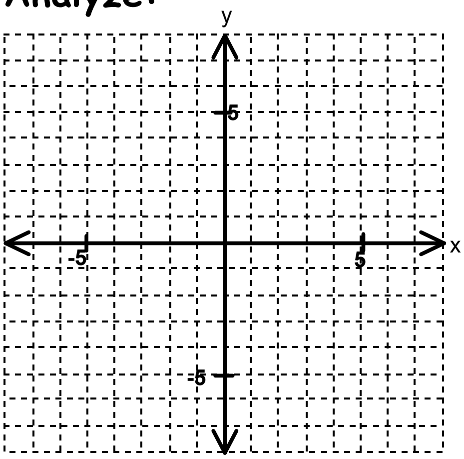
Reduce to 1 variable: ________________

Optimize and Verify:

Conclusion:

3. A rectangle is bounded by the *x*-axis and the semi-circle $y=\sqrt{25-x^2}$. What length and width should the rectangle have so that its area is a maximum?

Analyze:



Primary Equation: ____________

Reduce to 1 variable: _______________

Feasible Domain:

Optimize and Verify:

Conclusion:

4. The US Post Office will accept rectangular boxes only if the sum of the length and girth (twice the width plus twice the height) is at most 72 inches. What are the dimensions of the box of maximum volume the Post Office will accept? (You may assume that the width and height are equal.)

Analyze:

Primary Equation: ___________________ **Feasible Domain:**

Reduce to 1 variable: ________________

Optimize and Verify:

Conclusion:

3.9: Differentials

When you finish your homework you should be able to…

- π Understand the concept of a tangent line approximation
- π Compare the value of the differential, dy, with the actual change in y, Δy.
- π Estimate a propagated error using a differential.
- π Find the differential of a function using differentiation formulas.

Let f be a function that is ___________________ at c. The equation for the tangent line at point (c, f(c)) is:

____________________________ or ____________________________

We call this the **tangent line approximation** (or linear approximation) **of f at c**.

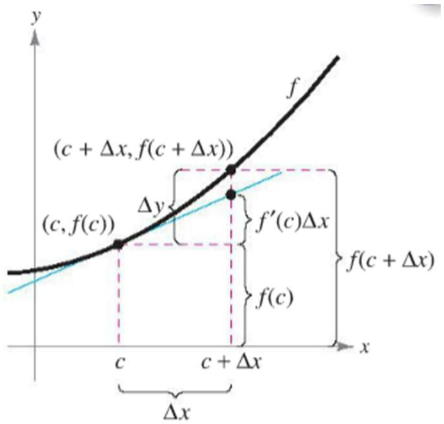
Example 1. Find a tangent line approximation T(x) to the graph $f(x) = x^2$, at $(2,4)$.

Use a graphing calculator to complete the chart below:

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$					
$T(x) =$ __________					

So at values _________ _________ to ______, _______ provides a good

__________________ of _______.



When Δx is small, $\Delta y = f(c + \Delta x) - f(c)$ is approximated by $f'(c)\Delta x$.

When the tangent line to the graph of f at the point (c,f(c))

$$y = f(c) + f'(c)(x - c)$$

is used to approximate the graph of f, (x – c) is called the *change in x*, and is denoted by Δx. When Δx is small, the change in y (or Δy) can be approximated by:

$$\Delta y = f(c + \Delta x) - f(c)$$

We denote Δx, as *dx* and call *dx* the **differential of x**. The expression $f'(x)dx$ is denoted *dy* and is called the **differential of y**.

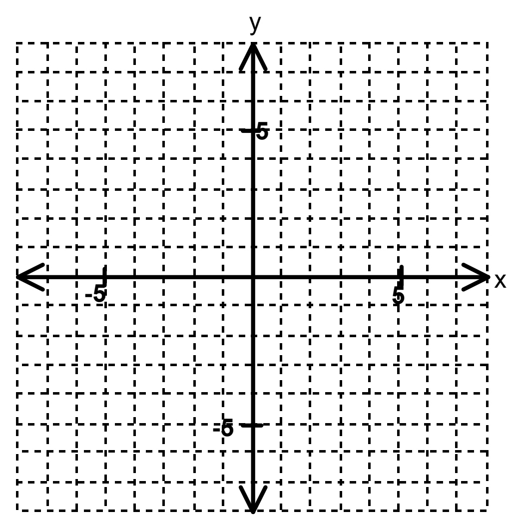
Definition of Differentials

Let $y = f(x)$ represent a function that is differentiable on an open interval containing x. The **differential of x** (denoted by dx) is any nonzero real number. The **differential of y** (denoted by dy) is

$$dy = f'(x)\,dx.$$

Example 2. Let $f(x) = 6 - 2x^2$

Compare Δy and *dy* at x = -2 and $\Delta x = dx = 0.1.$



Error Propagation

$$\underbrace{f(x+\overbrace{\Delta x}^{\text{Measurement error}})}_{\text{Exact value}} - \underbrace{f(x)}_{\text{Measured value}} = \overbrace{\Delta y}^{\text{Propagated error}}$$

Calculating Differentials:

Differential Formulas

Let u and v be differentiable functions of x.

Constant multiple:	$d[cu] = c\,du$
Sum or difference:	$d[u \pm v] = du \pm dv$
Product:	$d[uv] = u\,dv + v\,du$
Quotient:	$d\left[\frac{u}{v}\right] = \frac{v\,du - u\,dv}{v^2}$

Example 3: Find the differentials

	Derivative	Differential
1. $y = x^2$		
2. $y = \sqrt{x}$		
3. $y = -2cosx$		
4. $y = x\sqrt{1-x^2}$		

Example 4. The measurements of the base and altitude of a triangle are found to be 36 and 50 centimeters, respectively. The possible error in each measurement is 0.25 centimeter. Use differentials to approximate the possible propagated error in computing the area of the triangle.

Example 5. A surveyor standing 50 feet from the base of a large tree measures the angle of elevation to the top of the tree as 71.5°. How accurately must the angle be measured if the percent error in estimating the height of the tree is to be less than 6%?

Differentials can be used to approximate function values. To do this, we use the formula:

$$f(x + \Delta x) \approx f(x) + dy = f(x) + f'(x)dx$$

Example 6: Use differentials to approximate the following:

a. $\sqrt[3]{28}$

b. $\sin(0.1)$

4.1: Antiderivatives

When you finish your homework you should be able to…

- π Write the general solution of a differential equation
- π Use indefinite integral notation for antiderivatives
- π Use basic integration rules to find antiderivatives
- π Find a particular solution of a differential equation

Warm-up: For each derivative, describe an original function F.

a. $F'(x) = 2x$

d. $F'(x) = \sec^2 x$

b. $F'(x) = x^3$

e. $F'(x) = \sin x$

c. $F'(x) = \dfrac{1}{x^2}$

f. $F'(x) = 6$

DEFINITION OF ANTIDERIVATIVE

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Why is F called **an** antiderivative of f, rather than **the** antiderivative of f?

THEOREM: REPRESENTATION OF ANTIDERIVATIVES

If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.

NOTATION:

Example 1: Verify the statement by showing that the derivative of the right side equals the integrand of the left side.

$$\int \left(8x^3 + \frac{1}{2x^2}\right) dx = 2x^4 - \frac{1}{2x} + C$$

Example 2: Find the general solution of the differential equation.

a. $\frac{dy}{dx} = 2x^{-3}$

b. $\frac{dr}{d\theta} = \pi$

BASIC INTEGRATION RULES

Differentiation Formula	Integration Formula
$\frac{d}{dx}[C]=$	$\int 0dx=$
$\frac{d}{dx}[kx]=$	$\int kdx=$
$\frac{d}{dx}[kf(x)]=$	$\int kf(x)dx=$
$\frac{d}{dx}[f(x)\pm g(x)]=$	$\int [f(x)\pm g(x)]dx=$
$\frac{d}{dx}[x^n]=$	$\int x^n dx=$
$\frac{d}{dx}[\sin x]=$	$\int \cos x dx=$
$\frac{d}{dx}[\cos x]=$	$\int \sin x dx=$
$\frac{d}{dx}[\tan x]=$	$\int \sec^2 x dx=$
$\frac{d}{dx}[\sec x]=$	$\int \sec x \tan x dx=$
$\frac{d}{dx}[\cot x]=$	$\int \csc^2 x dx=$
$\frac{d}{dx}[\csc x]=$	$\int \csc x \cot x dx=$

Example 3: Find the indefinite integral and check the result by differentiation.

a. $\int (16 - x)dx$

b. $\int \frac{\sqrt[5]{x^3} - 2x}{\sqrt{x}}dx$

c. $\int (3x-4)^3 dx$

d. $\int (1-u)\sqrt{u}\,du$

e. $\int \sec t(\tan t - \sec t)dt$

f. $\int (4\theta - \csc^2 \theta)d\theta$

g. $\int \frac{1}{1-\sin x}dx$

INITIAL CONDITIONS AND PARTICULAR SOLUTIONS

You have already seen that the equation $y = \int f(x)dx$ has ___________________

solutions, each differing from each other by a ____________________________.

This means that the graphs of any two _____________________________ of f are

_____________________________ translations of each other.

In many applications of integration, you are given enough information to determine a ______________________________ solution. To do this, you need only know the value of $y = F(x)$ for one value of x. This information is called an

______________________________ condition.

How do the following differ?

x^2 versus $y = x^2$

How about:

$y = x^2$ versus $\dfrac{dy}{dx} = x^2$

Example 4: Solve the differential equation.

a. $g'(x) = 6x^2, \;\; g(0) = -1$

b. $f''(x) = \sin x, \; f'(0) = 1, \; f(0) = 6$

4.2: Area

When you finish your homework you should be able to…

- π Use Sigma Notation to Write and evaluate a Sum
- π Understand the Concept of Area
- π Approximate the Area of a Plane Region
- π Find the Area of a Plane Region Using Limits

Warm-up: Evaluate the following limits.

a. $\lim\limits_{x\to\infty}\dfrac{25x^2-5x+3}{x^2}$

b. $\lim\limits_{x\to\infty}\left(\dfrac{2x^3+6x^2}{3x^3}+10\right)$

SIGMA NOTATION

> The sum of n terms $a_1,\ldots,a_n$ is written as $\sum_{i=1}^{n}a_i=a_1+a_2+a_3+\cdots+a_n$ where i is the **index of summation**, a_i is the ***i*****th term** of the sum, and the **upper and lower bounds of summation** are n and 1.

Example 1: Find the sum.

a. $\sum\limits_{i=1}^{5}i$

b. $\sum\limits_{i=1}^{4}i^2$

c. $\sum\limits_{i=1}^{3}\sqrt{i}$

Summation Properties

1. $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$	2. $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Example 2: Find the sum.

a. $\sum_{i=1}^{4} 2$

b. $\sum_{i=1}^{3}(1-i^2)$

Theorem: Summation Formulas (These formulas will be provided for the exam)

1. $\sum_{i=1}^{n} c = cn$	2. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$	4. $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Example 3: Evaluate the following sums.

1. $\sum_{i=1}^{n}(3i-i^2)$

2. $\sum_{i=1}^{n}(6i+4i^3)$

Area

In Euclidean geometry, the simplest type of plane region is a ________________. The definition for the area of a rectangle is __________.From this definition, you can develop formulas for the areas of many other plane regions such as triangles. To determine the area of a triangle, you can form a _________________ whose area is _____________ that of the __________________. Once you know how to find the area of a triangle, you can determine the area of any _______________ by subdividing the polygon into _________________ regions.

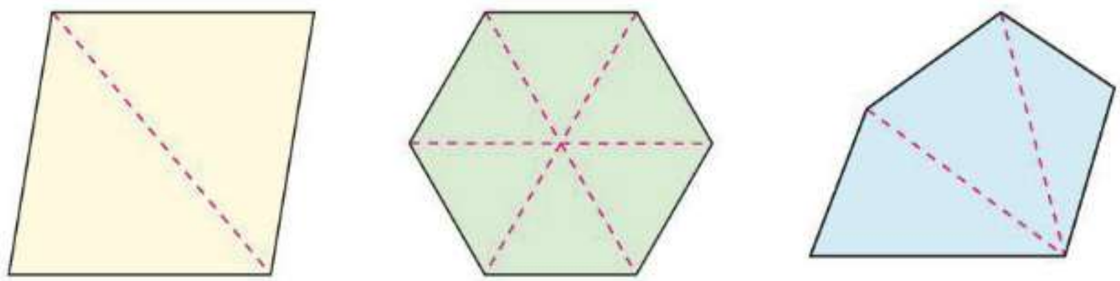
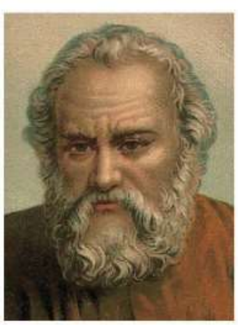


Figure 5.6

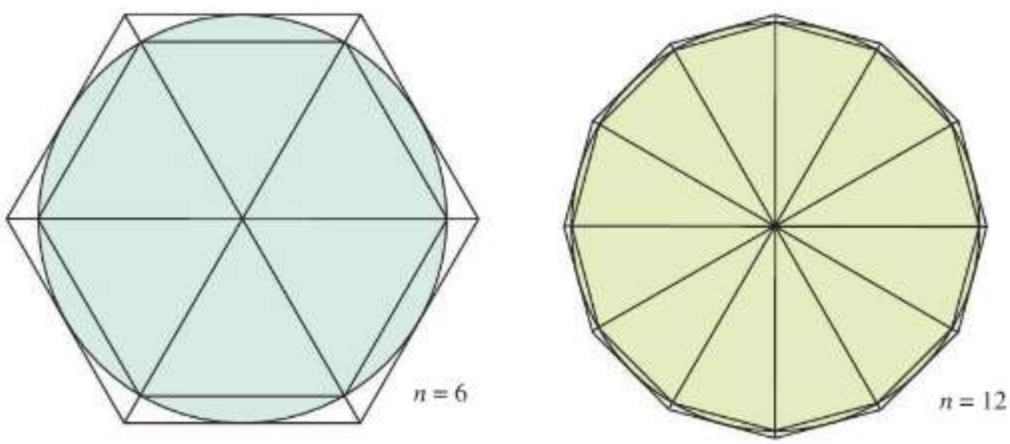
Finding the areas of regions other than polygons is more difficult. The ancient Greeks were able to determine formulas for the areas of some general regions by the ________________ method. Essentially, the method is a ________________ process in which the area is _____________________ between two polygons—one __________________ in the region and one ________________________ about the region.



ARCHIMEDES (287–212 B.C.)

Archimedes used the method of exhaustion to derive formulas for the areas of ellipses, parabolic segments, and sectors of a spiral. He is considered to have been the greatest applied mathematician of antiquity.

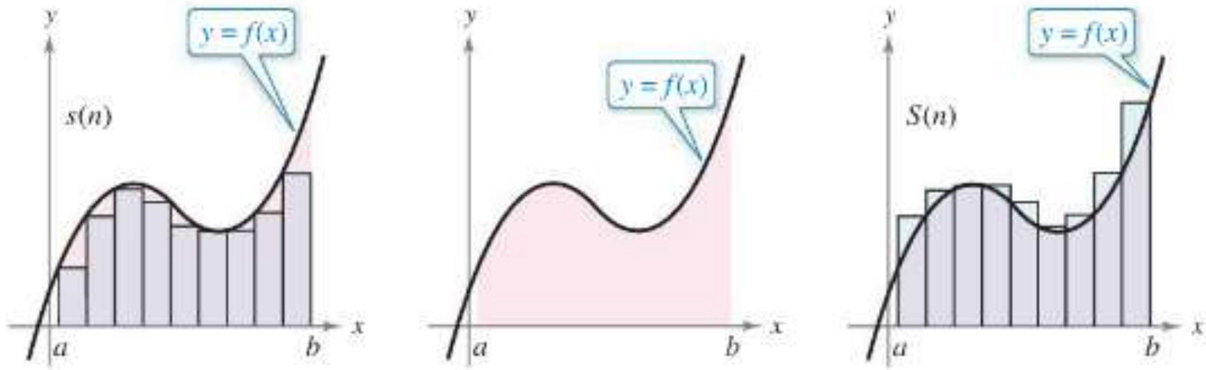
See LarsonCalculus.com to read more of this biography.



The exhaustion method for finding the area of a circular region

Figure 5.7

$s(n) \leq (\text{Area of region}) \leq S(n)$



Area of inscribed rectangles is less than area of region.

Area of region

Area of circumscribed rectangles is greater than area of region.

Figure 5.11

Theorem: Limits of the Lower and Upper Sums

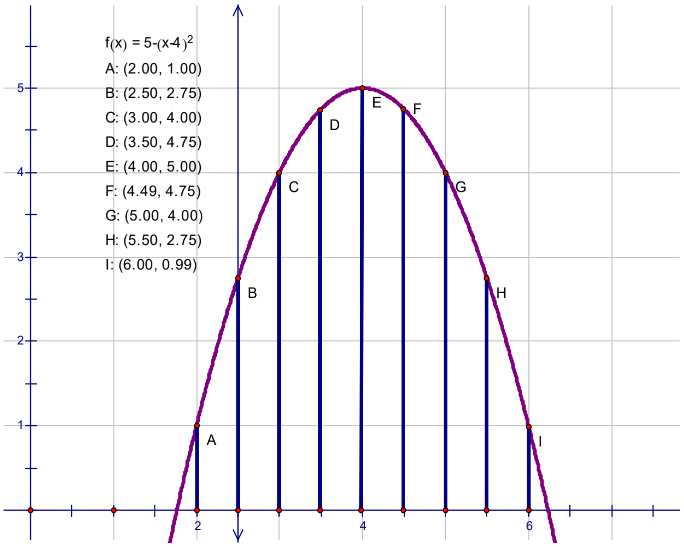
Let f be continuous and nonnegative on the interval $[a,b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty} \sum_{i=1}^{n} f(m_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(M_i)\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

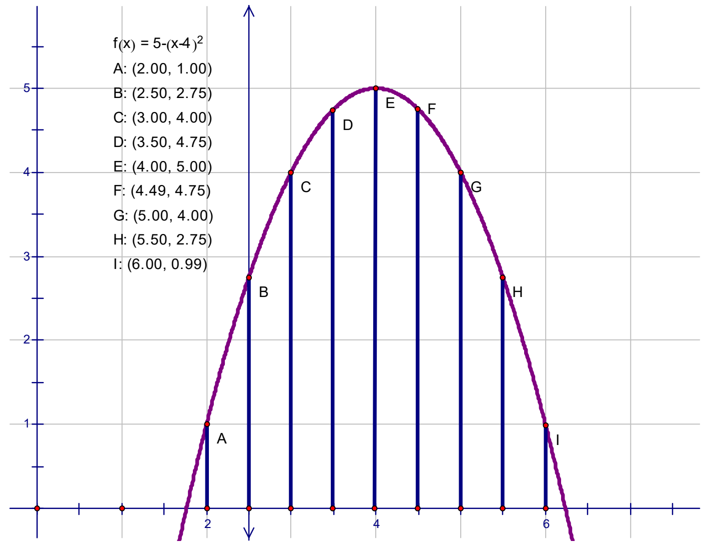
where $\Delta x = \dfrac{b-a}{n}$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Example 4:

a. Find the upper sum from $x = 2$ to $x = 6$.



b. Find the lower sum from $x = 2$ to $x = 6$.



Definition of an Area in the Plane

Let f be continuous and nonnegative on the interval $[a,b]$. The area of the region bounded by the graph of f, the *x*-axis, and the vertical lines $x=a$ and $x=b$ is

Area $= \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x, \qquad x_{i-1} \le c_i \le x_i \qquad$ where $\Delta x = \frac{b-a}{n}$,

right endpoint: $c_i = a + i\Delta x$, left endpoint: $c_i = a + (i-1)\Delta x$

Example 5: Find the area of the region bounded by the graph $f(x) = x^3$, the *x*-axis, and the vertical lines $x=0$ and $x=1$.

4.3: Riemann Sums and Definite Integrals

When you finish your homework you should be able to…

- π Understand the definition of a Riemann sum.
- π Evaluate a definite integral using limits.
- π Evaluate a definite integral using properties of definite integrals.

$$\sum_{i=1}^{n} c = nc \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

1. Evaluate $\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i$ over the region bounded by the graphs of $f(x)=\sqrt[3]{x},\ y=0,\ x=0,\ x=1$. Hint: Let $c_i = {i^3}/{n^3}$ and recall that the width of each interval is $\Delta x_i = \frac{i^3}{n^3}-\frac{(i-1)^3}{n^3}$.

Definition of Riemann Sum

Let f be defined on the closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by

$$a = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_{n-1} < x_n = b$$

where Δx_i is the width of the ith subinterval

$[x_{i-1}, x_i]$. ith subinterval

If c_i is *any* point in the ith subinterval, then the sum

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i, \quad x_{i-1} \le c_i \le x_i$$

is called a **Riemann sum** of f for the partition Δ. (The sums in Section 4.2 are examples of Riemann sums, but there are more general Riemann sums than those covered there.)

____________ is called the ____________ of ____________, the width of the largest subinterval. As _________________, ______________.

Definition of Definite Integral

If f is defined on the closed interval $[a, b]$ and the limit of Riemann sums over partitions Δ

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i$$

exists (as described above), then f is said to be **integrable** on $[a, b]$ and the limit is denoted by

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\,\Delta x_i = \int_a^b f(x)\,dx.$$

The limit is called the **definite integral** of f from a to b. The number a is the **lower limit** of integration, and the number b is the **upper limit** of integration.

2. Evaluate the definite integral by the limit definition.

$$\int_{1}^{6} \left(2x^2 + 1\right) dx$$

Theorem 4.4 Continuity Implies Integrability

If a function f is ____________________ on the closed interval ______, then f is _______________ on [a,b]. That is _____________ exists.

THEOREM 4.5 The Definite Integral as the Area of a Region

If f is continuous and nonnegative on the closed interval $[a, b]$, then the area of the region bounded by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is

$$\text{Area} = \int_a^b f(x)\,dx.$$

(See Figure 4.22.)

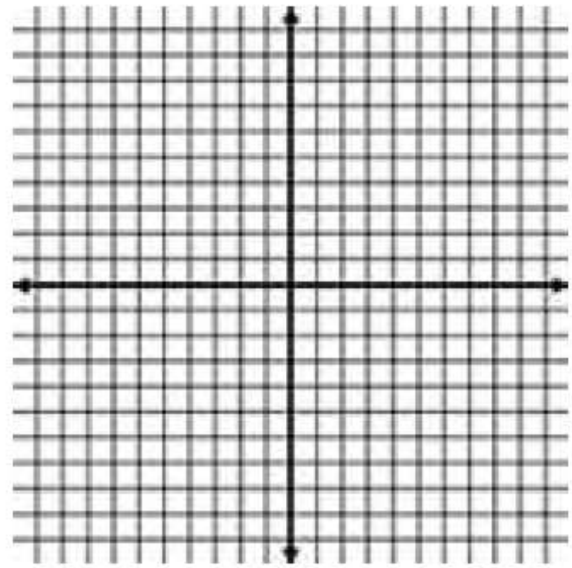
3. Write the limit as a definite integral on the interval $[a,b]$ where c_i is any point on the ith interval.

a. $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} (8c_i + 15)\Delta x_i, \quad [2,6]$

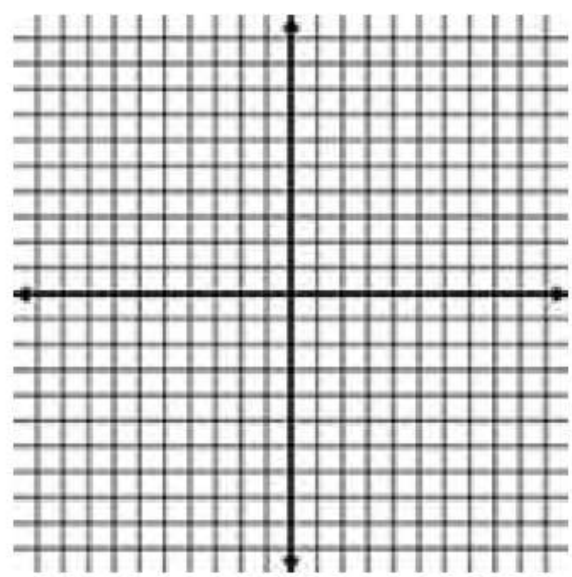
b. $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} 5c_i\sqrt{c_i^2 + 2}\Delta x_i, \quad [0,12]$

4. Sketch the region whose area is given by the definite integral. Then use a geometric formula to evaluate the integral.

a. $\int_0^3 3x\,dx$



b. $\int_{-4}^{4} \sqrt{16-x^2}\,dx$



Definitions of Two Special Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.

2. If f is integrable on $[a, b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

5. Given $\int_0^3 f(x)dx = 4$ and $\int_3^6 f(x)dx = -1$, evaluate

a. $\int_0^6 f(x)dx$

b. $\int_6^3 f(x)dx$

c. $\int_3^3 f(x)dx$

4.4: The Fundamental Theorem of Calculus

When you finish your homework you should be able to…

- π Evaluate a definite integral using the Fundamental Theorem of Calculus.
- π Understand and use the Mean Value Theorem for Integrals.
- π Find the average value of a function over a closed interval.
- π Understand and use the Second Fundamental Theorem of Calculus.
- π Understand and use the Net Change Theorem.

Theorem: The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a,b]$ and F is an antiderivative of f on the interval $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Guidelines for Using the Fundamental Theorem of Calculus

1. Provided you can find an antiderivative of f, you now have a way to evaluate a definite integral without having to use the limit of a sum.
2. When applying the Fundamental Theorem of Calculus, the following notation is convenient:

$$\int_a^b f(x)\,dx = F(x)\Big|_{x=a}^{x=b}$$
$$= F(b) - F(a)$$

3. It is not necessary to include a constant of integration C in the antiderivative because

$$\int_a^b f(x)\,dx = \left[F(x) + C\right|_{x=a}^{x=b}$$
$$= \left[F(b) + C\right] - \left[F(a) + C\right]$$
$$= F(b) - F(a)$$

Helpful hints:

Rational Functions must be written as the sum/difference of functions (neg. exponents are ok)

$$\int_a^b \frac{5\sqrt{x} - x^2 + 2}{x^2}\,dx = \int_a^b (5x^{-3/2} - 1 + 2x^{-2})\,dx$$

Products of functions (not a constant times a function) must be multiplied out.

$$\int_a^b (x+3)(2x-1)\,dx =$$

Trig functions:

Use identities – try to change everything to sines and cosines

Pythagorean Conjugates - generate a difference of squares situation so that you can change the denominator from 2 terms to 1.

$$\int_a^b \frac{1}{1+\sin x}\,dx =$$

Example 1. Evaluate the definite integral.

a. $\int_{-2}^{6} 6dx$

$f(x) =$ ____________

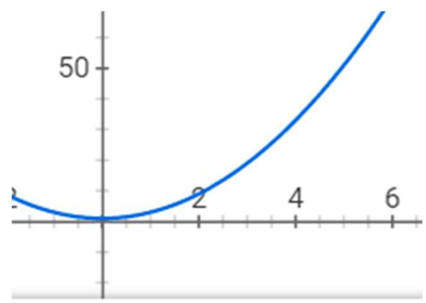
$F(x) =$ ____________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

$F(a) =$ ______ $=$ _____

b. $\int_{1}^{6} \left(2x^2 + 1\right)dx$



$f(x) =$ ____________

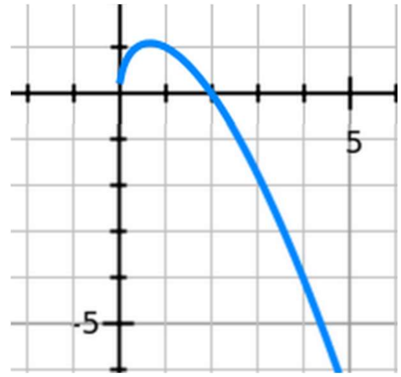
$F(x) =$ ____________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

$F(a) =$ ______ $=$ _____

c. $\int_0^2 (2-t)\sqrt{t}\,dt$



$f(t) =$ ______________

$F(t) =$ ______________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

$F(a) =$ ______ $=$ _____

d. $\int_1^4 (2v+5)^3\,dv$

$f(v) =$ ______________

$F(v) =$ ______________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

$F(a) =$ ______ $=$ _____

Example 2: Evaluate the definite integral.

a. $\int_{\pi/6}^{\pi/3} \frac{1}{1-\cos x}\,dx =$

$f(x) =$ ______________

$F(x) =$ ______________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

$F(a) =$ ______ $=$ _____

b. $\int_2^6 (x^2 - 8)^2 \, dx =$

$f(x) =$ ____________

$F(x) =$ ____________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

$F(a) =$ ______ $=$ _____

Example 3. Find the area of the region bounded by the graphs of the equations.

$$y = 1 + \sqrt[3]{x}, \quad x = 0, \quad x = 8, \quad y = 0.$$

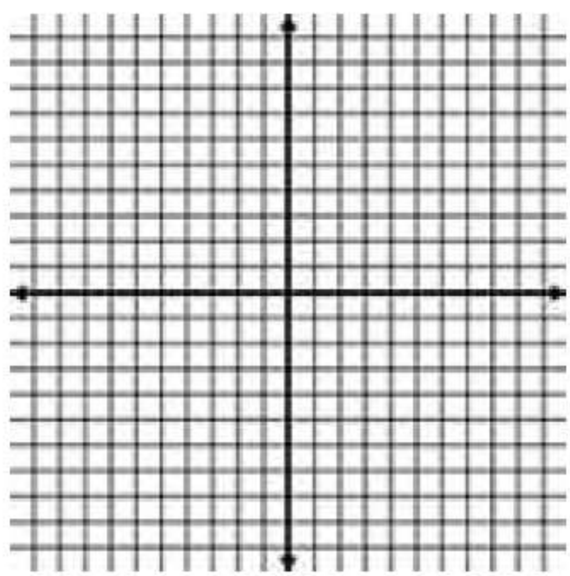
$f(x) =$ ____________

$F(x) =$ ____________

$a =$ _____ $b =$ _____

$F(b) =$ ______ $=$ _____

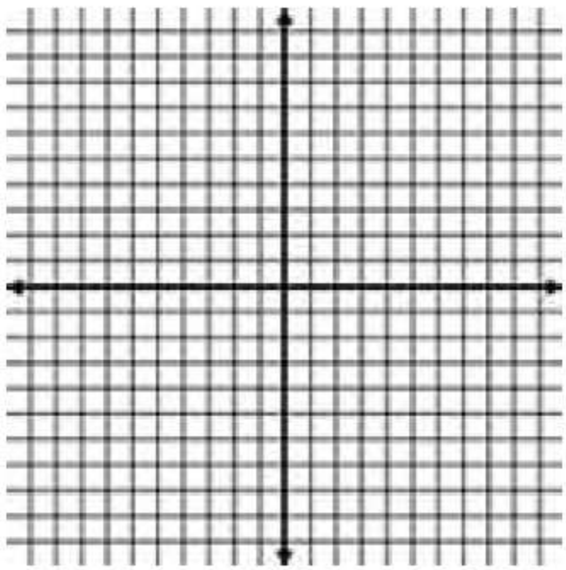
$F(a) =$ ______ $=$ _____



THE MEAN VALUE THEOREM FOR INTEGRALS

If f is continuous on the closed interval $[a,b]$, then there exists a number c in the closed interval $[a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b-a)$$



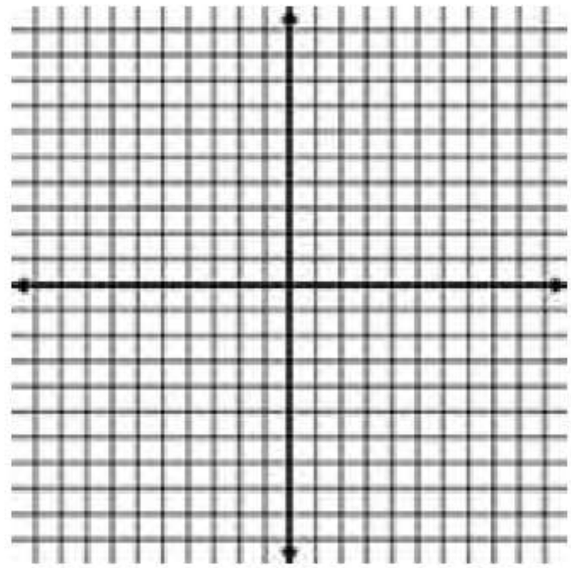
Definition of the Average Value of a Function on an Interval

If f is integrable on the closed interval $[a,b]$, then the **average value** of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

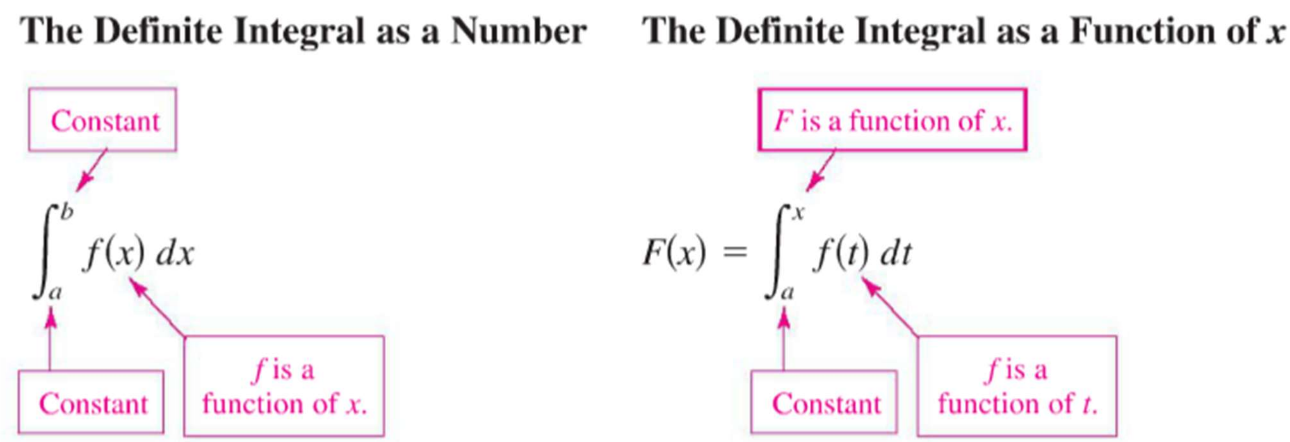
If you isolate ________ in the ____________ ___________ ________________ for _________________, the other side of the ________________ is the ______________ _____________ of the function.

Example 3. Find the value(s) of c guaranteed by the Mean Value Theorem for Integrals for the function $f(x) = \cos x, \quad \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$



Example 4. Find the average value of the function $f(x)=\dfrac{4(x^2+1)}{x^2}, \quad [1,3]$ and all the values of *x* in the interval for which the function equals its average value.

The second fundamental theorem of calculus:



THEOREM 4.11 The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x).$$

Consider

$$F(x) = \int_3^x (1-t)\,dt$$

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Example 5.

a. Find $\frac{d}{dx}\int_0^x (t^2+7)dt$

b. Find $\frac{d}{dx}\int_0^x \frac{t^4}{1+t^6}dt$

$$\frac{d}{dx}\int_{a}^{u=g(x)} f(t)dt =$$

c. Find $\frac{d}{dx}\int_{1}^{x^2} \ln t dt$

d. Find $\frac{d}{dx}\int_0^{\cos x}\sqrt{1-t^2}\,dt$

The Net Change Theorem:

The definite integral of the rate of change of quantity $F'(x)$ gives the total change, or **net change,** in that quantity on the interval $[a, b]$.

$$\int_a^b F'(x)\,dx = F(b) - F(a) \qquad \text{Net change of } F$$

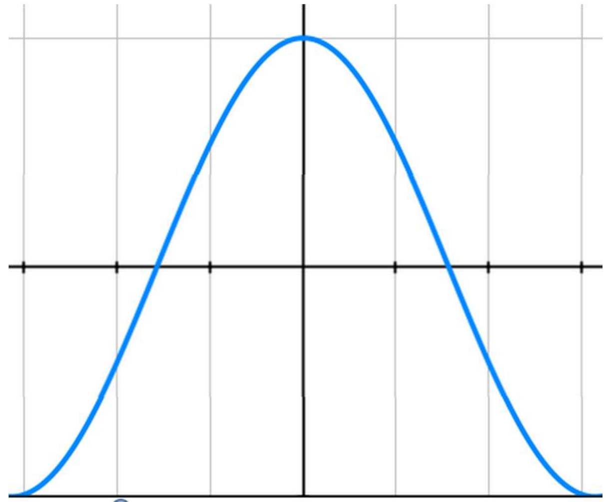
Example 6: At 1:00PM, oil begins leaking from a tank at a rate of $(4+0.75t)$ gallons per hour.

a. How much oil is lost from 1:00PM-4:00PM?

b. How much oil is lost from 4:00PM-7:00PM?

c. What did you notice?

Example 7. Consider the integral $\int_0^{\pi} \cos x dx$



What if we wanted to find the total area?

4.5: Integration by Substitution

When you finish your homework you should be able to…

- π Use pattern recognition to find an indefinite integral.
- π Use a change of variables to find an indefinite integral.
- π Use the General Power Rule for Integration to find an indefinite integral.
- π Use a change of variables to evaluate a definite integral.
- π Evaluate a definite integral involving an even or odd function.

Warm-up: Find the indefinite integral or evaluate the definite integral.

a. $\int \sec x \tan x dx$

b. $\int \left(\sin^2 x + \cos^2 x\right) dx$

c. $\int_3^4 \frac{5x^2 - 7x - 6}{x - 2} dx$

When trying to integrate ________________ functions, we can use a method called "________________of ___________________________", or, as we gain ore experience, we can use __________________________ recognition to integrate the composite function without as much extra work. Just like with the Chain Rule in differentiation, the more you ______________________, the easier it is to "___________________" the derivative without taking the time to write out all the steps.

Antidifferentiation of a Composite Function

Let *g* be a function whose range is an interval *I*, and let *f* be a function that is continuous on *I*. If *g* is differentiable on its domain and F is an antiderivative of *f* on *I*, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

By substitution, if we let $u = g(x)$ then $du = g'(x)$ and

$$\int f(u)du = F(u) + C$$

In order to use u-substitution, you must recognize that you are trying to integrate a composite function. It is important for you to be able to decompose the composite function.

$\int f\left(g\left(x\right)\right)g'\left(x\right)dx = F\left(x\right)+C$ $\qquad$ $\int f\left(u\right)du = F\left(u\right)+C$

Example 1. Find the indefinite integral.

a. $\int (x^2-3)^2(2x)dx$

b. $\int 3x^2\sqrt{x^3+1}dx$

$\int f(g(x))g'(x)\,dx = F(x) + C$ $\int f(u)\,du = F(u) + C$

Many integrands contain the essential part of $g'(x)$ but are missing a constant multiple. How will we deal with that when it happens?

Example 2. Find the indefinite integral.

$$\int x^2 \sin(x^3)dx$$

Change of Variables

We can be more formal and rewrite the integral in terms of another variable (usually we use the variable *u* and therefore we refer to this as *u-substitution)*.

Guidelines for Making a Change of Variables

1. Choose a substitution u. Usually, it is best to choose the *inner* part of a composite function, such as a quantity raised to a power.
2. Compute $\frac{d}{dx}(u) = \boxed{} \to dx = \frac{du}{\boxed{}}$.

3. Rewrite the integral in terms of the variable u.

4. Find the resulting integral in terms of u.

5. Back substitute u by the original expression to obtain an antiderivative in terms of x (or whatever your original variable was).

6. Check your answers by differentiating.

$$\int f\left(g(x)\right)g'(x)\,dx = F\left[g(x)\right]+C \qquad\qquad \int f(u)\,du = F(u)+C$$

Example 3. Evaluate the following:

a. $\int \sqrt{t+1}\,dt$

b. $\int t\sqrt{t+1}\,dt$

$$\int f\left(g\left(x\right)\right)g'\left(x\right)dx = F\left[g\left(x\right)\right]+C \qquad \int f\left(u\right)du = F\left(u\right)+C$$

c. $\displaystyle\int \frac{x^4+2}{\left(x^5+10x\right)^5}\,dx$

$\int f(g(x))g'(x)\,dx = F[g(x)] + C$ $\int f(u)\,du = F(u) + C$

d. $\int \sin^2 2x \cos 2x dx$

e. $\int \sin^2 2x dx$

Shifting Rule for Integration:

To evaluate $f(x+a)dx$ first evaluate $\int f(u)du$, then substitute $x+a$ for u:

$$f(x+a)dx = F(x+a)+C, \text{ where } F(u)=\int f(u)du.$$

There are substitutions that are so useful, that they are worth noting explicitly. We have already used them in the previous examples:

Scaling Rule for Integration:

To evaluate $f(bx)dx$ evaluate $\int f(u)du$, divide by b and substitute for bx for u:

$$f(bx)dx = \frac{1}{b}F(bx)+C, \text{ where } F(u)=\int f(u)du.$$

Theorem: General Power Rule for Integration:

If g is a differentiable function of x, then

$$\int [g(x)]^n g'(x)dx = \frac{[g(x)]^{n+1}}{n+1} + C, \; n \neq -1.$$

Equivalently, if $u = g(x)$ then

$$\int (u)^n du = \frac{u^{n+1}}{n+1} + C, \; \text{n} \neq -1.$$

Theorem: Change of Variable for Definite Integrals

If the function $u = g(x)$ has a continuous derivative on the closed interval [a,b] and f is continuous on the range of g, then

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

Example 5: Evaluate the definite integrals:

a. $\displaystyle\int_1^3 \frac{3x}{(x^2+5)^2} dx$

b. $\int_0^{\pi/6} x^2 \tan^2 x^3 dx$

Theorem: Integration of Even and Odd Functions

Let *f* be integrable on the closed interval [a,-a].

1. If *f* is an even function, then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

2. If *f* is an odd function, then $\int_{-a}^{a} f(x)dx = 0$

Example 5: Evaluate the definite integral.

a. $\int_{-\pi/2}^{\pi/2} \sin x \cos x dx$

b. $\int_{-2}^{2} \left(x^4 - 3x^2\right) dx$

5.1: The Natural Logarithmic Function: Differentiation

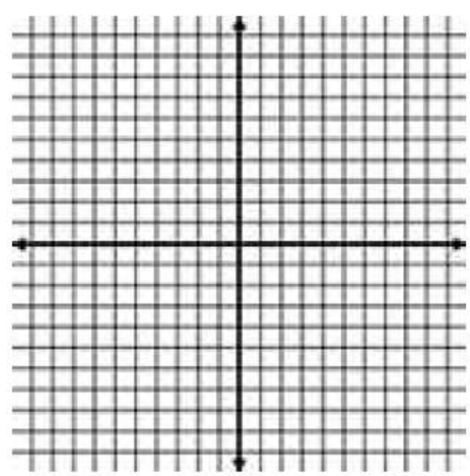
When you are done with your homework you should be able to…

- π Develop and use properties of the natural logarithmic function
- π Understand the definition of the number e
- π Find derivatives involving the natural logarithmic function

Warm-up:

1. Use the limit definition of the derivative to find the derivative of $f(x)=\frac{3}{x}$ with respect to *x*.

2. Graph $y=\ln x$.



DEFINITION OF THE LOGARITHMIC FUNCTION BASE e

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the **logarithmic function to the base e** is defined as

$\log_a x =$ ________________ .

The number e:

The number e can be defined as a ______________; specifically, $\lim_{v \to 0}(1+v)^{1/v}$.

The natural logarithmic function and the natural exponential function are __________________ of each other.

PROPERTIES OF INVERSE FUNCTIONS

1. $y = e^x$ iff ________________
2. $e^{\ln x} =$ _____, for $x > 0$
3. $\ln e^x =$ _______, for all x

PROPERTIES OF NATURAL LOGS

1. $\ln 1 =$ _____
2. $\ln e =$ _____
3. $\ln xy =$ ______________
4. $\ln \frac{x}{y} =$ ______________
5. $\ln x^n =$ ______________

The inverse relationship between the natural logarithmic function and the natural exponential function can summarized as follows:

Example 1: Condense the following logarithmic expressions.

a. $\ln(x+8)-\left[\ln(x-2)-5\ln(x)\right]$

b. $\frac{1}{2}\ln x+8\ln z-\ln y$

Example 2: Expand the following logarithmic expressions.

a. $\ln \sqrt[4]{\left(\frac{x^2-1}{2x+5}\right)^3}$

b. $\left(\frac{1-\cos x}{\cos 2x}\right)^5$

Example 3: Solve the following equations. Give the **exact result** and then round to 3 decimal places.

a. $\ln(x-2)+\ln(x+2)=16$

b. $\dfrac{1}{2}=30e^{3t}$

DERIVATIVE OF THE NATURAL LOGARITHMIC FUNCTION

THEOREM: DERIVATIVES OF THE NATURAL LOGARITHMIC FUNCTION

Let u be a differentiable function of x such that $u \neq 0$, then

1. $\frac{d}{dx}[\ln x] =$ ________

2. $\frac{d}{dx}[\ln u] =$ ________________

Example 4: Find the derivative with respect to x.

a. $y = \ln 5x$

b. $f(x) = \ln(1-2x)$

c. $f(x) = \ln x^x$

d. $y=\ln\sqrt{\dfrac{x^2+1}{1-x}}$

e. $y=\ln\cos^2 x$

f. $\ln xy + x^2 - y = 10$

THEOREM: DERIVATIVE INVOLVING ABSOLUTE VALUE

Let u be a differentiable function of x such that $u \neq 0$, then

$$\frac{d}{dx}\left[\ln|u|\right] = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

Proof:

Example 5: Find the derivative of the function with respect to *x*.

$$f(x) = \ln\left|\sec x + \tan x\right|$$

Example 6: Differentiate the following functions with respect to *x*.

a. $f(x) = \sqrt{x^2(x+1)(x+2)}$

b. $y=\left(\dfrac{x^2-4}{x^2+4}\right)^{2/3}$

Example 7: Find an equation of the tangent line of the function $f(x) = \sin 2x \ln x^2$ at the point $(1,0)$.

5.2: The Natural Logarithmic Function: Integration

When you are done with your homework you should be able to…

- π Use the Log Rule for Integration to integrate a rational function
- π Integrate trigonometric functions

Warm-up:

1. Differentiate the following functions with respect to *x*.

   a. $y = x\ln 5x$

   b. $\ln(xy) = \ln(x+y)$.

THEOREM: LOG RULE FOR INTEGRATION

Let u be a differentiable function of x.

1. $\int \frac{1}{x} dx = \ln|x| + C$

2. $\int \frac{1}{u} dx = \ln|u| + C$

Example 1: Find or evaluate the integral.

a. $\int \frac{10}{x} dx$

b. $\int \frac{x^2}{\sqrt{5-x^3}}dx$

c. $\int \frac{x}{\sqrt{1-x^2}}dx$

d. $\int_{e}^{e^2} \frac{dx}{x \ln x}$

e. $\int_{1}^{e} \frac{\left(1+\ln x\right)^2 dx}{x}$

f. $\int \frac{1}{x^{2/3}\left(1+x^{1/3}\right)}dx$

g. $\int \frac{x^3-6x-20}{x+5}dx$

h. $\int \tan\theta d\theta$

i. $\int \cot\theta d\theta$

j. $\int \sec\theta d\theta$

k. $\int \csc\theta d\theta$

INTEGRALS OF THE SIX BASIC TRIGONOMETRIC FUNCTIONS

$\int \sin u du =$ ________________	$\int \cos u du =$ ________________
$\int \tan u du =$ ________________	$\int \cot u du =$ ________________
$\int \sec u du =$ ________________	$\int \csc u du =$ ________________

Example 2: Solve the differential equation.

a. $y' = \dfrac{x+1}{x-1}$

b. $r' = \theta \tan \theta^2$

Example 3: The demand equation for a product is $p = \dfrac{90,000}{400+3x}$. Find the average price on the interval $40 \le x \le 50$.

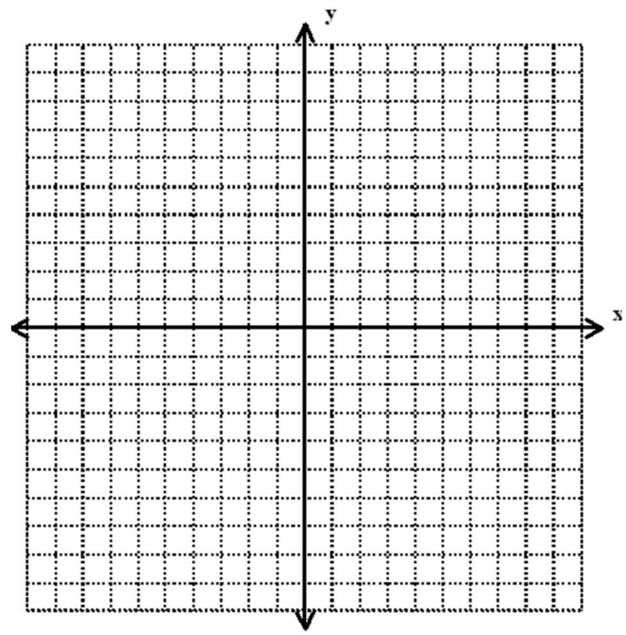
5.3: Inverse Functions

When you are done with your homework you should be able to…

- π Verify that one function is the inverse of another function
- π Determine whether a function has an inverse function
- π Find the derivative of an inverse function

Warm-Up:

1. Use the Horizontal Line Test to show that $f(x)=x^5+3$ is one-to-one.



2. Let $g(x)=\dfrac{1-2x}{x}$ and $h(x)=\dfrac{x}{x-2}$ find $g(h(x))$

Recall from Precalculus:

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Important Observations about inverse functions:

1. If g is the ____________________ function of f, then ______ is the inverse function of _______.
2. The ______________ of f^{-1} is equal to the _______________ of _____, and the _____________ of f^{-1} is equal to the _________________ of _____.
3. If a function has an inverse, the inverse is ________________.
4. $f^{-1}(f(x)) =$ ___________ and $f(f^{-1}(x)) =$ ______________.
5. A function has an inverse if and only if it is ____________ - ______ -__________.

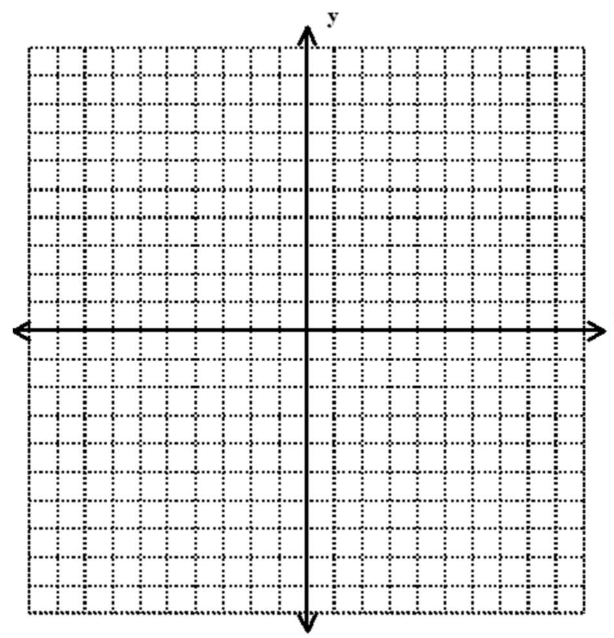
THEOREM 5.7 The Existence of an Inverse Function

1. A function has an inverse function if and only if it is one-to-one.
2. If f is strictly monotonic on its entire domain, then it is one-to-one and therefore has an inverse function.

Proof:

Example 1: Graph each function and verify that the following functions are inverses of each other.

$f(x) = 16 - x^2,\ x \le 0 \quad g(x) = -\sqrt{16 - x}$



GUIDELINES FOR FINDING AN INVERSE FUNCTION

1. Use Theorem 5.7 to determine whether the function $y = f(x)$ has an inverse function.
2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
4. Define the domain of f^{-1} as the range of f.
5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Example 2: Find f^{-1} of $f(x) = \dfrac{3x+2}{1-x}$.

Example 3: Prove that $f(x) = x^3 + 2x - 3$ has an inverse function (you do not need to find it).

Derivative of an Inverse Function:

THEOREM 5.8 Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true.

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is increasing on its domain, then f^{-1} is increasing on its domain.
3. If f is decreasing on its domain, then f^{-1} is decreasing on its domain.
4. If f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof:

THEOREM 5.9 The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))}, \quad f'(g(x)) \neq 0.$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proof:

Example 4. Let $f(x) = x^3 + 2x - 3$

a. What is the value of $f^{-1}(x)$ when x = 0?

b. What is the value of $f'(x)$ when x = 0?

c. What is the value of $(f^{-1})'(x)$ when x = 0?

d. What do you notice about the slopes of f and f^{-1}

Graphs of inverse functions have ___________________________ slopes.

5.4: Exponential Functions: Differentiation and Integration

When you are done with your homework you should be able to…

- π Develop properties of the natural exponential function
- π Differentiate natural exponential functions
- π Integrate natural exponential functions

Warm-up:

1. Differentiate the following functions with respect to *x*.

   a. $y = x^{5x}$

   b. $f\left(x\right) = \ln e^{\cos 2x}$.

DEFINITION: THE NATURAL EXPONENTIAL FUNCTION

The inverse function of the natural logarithmic function $f(x) = \ln x$ is called the **natural exponential function** and is denoted by

That is,

The inverse relationship between the natural logarithmic function and the natural exponential function can summarized as follows:

Example 1: Solve the following equations. Give the **exact result** and then round to 3 decimal places.

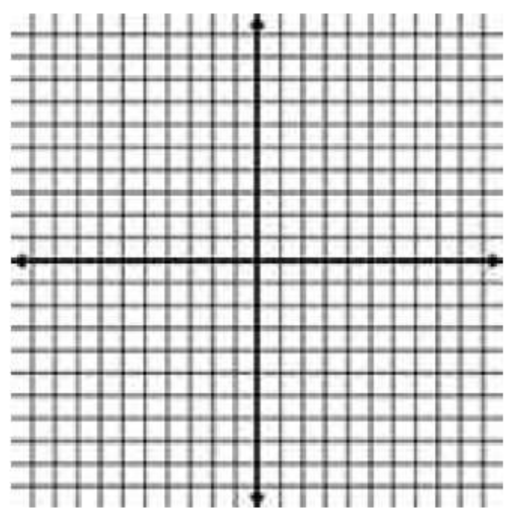
a. $e^{\ln 6x} = 20$

b. $\frac{5000}{1+e^{2x}}=2$

c. $\ln 8x = 3$

d. $e^{2x} - 2e^{x} - 8 = 0$

Example 2: Sketch the graph of $f\left(x\right) = 2e^{x-1}$ without using your graphing calculator.



DERIVATIVE OF THE NATURAL EXPONENTIAL FUNCTION

THEOREM: DERIVATIVES OF THE NATURAL EXPONENTIAL FUNCTION

Let u be a differentiable function of x.

1. $\frac{d}{dx}\left[e^x\right]=e^x$

2. $\frac{d}{dx}\left[e^u\right]=$ ____________________

Example 3: Find the derivative with respect to x.

a. $y=e^{5-x^3}$

b. $f(x) = xe^{3x}$

c. $y = \ln\frac{1+e^x}{1-e^x}$

d. $y=\dfrac{e^{x}-e^{-x}}{2}$

e. $e^{xy} + x^2 - y^2 = 10$

f. $F(x) = \int_0^{e^{2x}} \ln(t+1)dt$

Example 4: Find an equation of the tangent line of the function $1 + \ln xy = e^{x-y}$ at the point $(1,1)$.

Example 5: Find the extrema and points of inflection of the function

$$g(x)=\frac{1}{\sqrt{2\pi}}e^{-(x-3)^2/2}.$$

THEOREM: INTEGRATION RULES FOR NATURAL EXPONENTIAL FUNCTIONS

Let u be a differentiable function of x.

1. $\int e^x dx =$ ______________ 2. $\int e^u du =$ ______________

Example 6: Find the indefinite integrals and evaluate the definite integrals.

a. $\int e^{12x} dx$

b. $\int x^4 e^{1-x^5} dx$

c. $\int \frac{e^{2x}}{1+e^{2x}}\,dx$

d. $\int \frac{e^{2x}+2e^{x}+1}{e^{x}}\,dx$

e. $\int e^{\tan 2x} \sec^2 2x dx$

f. $\int_0^1 \frac{e^x}{5-e^x} dx$

Example 7: Solve the differential equation.

$$\frac{dy}{dx} = \left(e^{x} - e^{-x}\right)^{2} dx$$

Example 8: The median waiting time (in minutes) for people waiting for service in a convenience store is given by the solution of the equation $\int_0^x 0.3e^{-0.3t}\,dt = \frac{1}{2}$. Solve the equation.

5.5: Bases other than *e* and Applications

When you are done with your homework you should be able to…

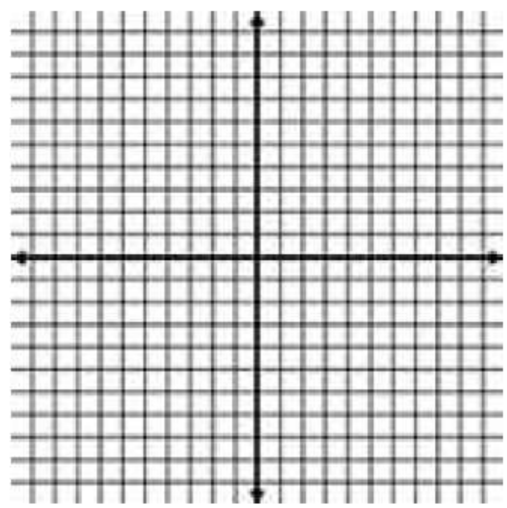
- π Define exponential functions that have bases other than e
- π Differentiate and integrate exponential functions that have bases other than e
- π Use exponential functions to model compound interest and exponential growth

Warm-up:

1. Evaluate the expression without using a calculator.

   a. $\log_{12} 144$ b. $\log_4 \frac{1}{16}$ c. $\log_{1/5} 25$

2. Sketch the graph of $f(x) = 2^x$ without using your graphing calculator.



DEFINITION: EXPONENTIAL FUNCTION TO BASE a

If a is a positive real number ($a \neq 1$) and x is any real number, then the **exponential function to the base a** is denoted by a^x and is defined by

$a^x = e^{(\ln a)x}$.

If $a = 1$, then $y = 1^x = 1$ is a constant function.

DEFINITION OF LOGARITHMIC FUNCTION TO BASE a

If a is a positive real number ($a \neq 1$) and x is any positive real number, then the **logarithmic function to the base *a*** is denoted by $\log_a x$ and is defined as $\log_a x = \dfrac{1}{\ln a} \ln x$.

PROPERTIES OF LOGS

1. $\log_a 1 =$ _____
2. $\log_a a =$ _____
3. $\log_a xy =$ ______________
4. $\log_a x^n =$ ______________

PROPERTIES OF INVERSE FUNCTIONS

1. $y = a^x$ iff ________________
2. $a^{\log_a a} =$ _____, for $x > 0$
3. $\log_a a^x =$ _______, for all x

THEOREM: DERIVATIVES FOR BASES OTHER THAN e

Let a be a positive real number ($a \neq 1$) and let u be a differentiable function of x.

1. $\dfrac{d}{dx}\left[a^x\right] =$ ______________
2. $\dfrac{d}{dx}\left[a^u\right] =$ ______________
3. $\dfrac{d}{dx}\left[\log_a x\right] =$ ______________
4. $\dfrac{d}{dx}\left[\log_a u\right] =$ ______________

Example 1: Differentiate the following functions with respect to *x*.

a. $y = 2^{-x}$

b. $f(x) = x\left(3^{8-x^2}\right)$

c. $f(t)=\dfrac{4^{2t}}{t}$

d. $g(t)=\log_2\left(t^2+7\right)^3$

e. $y = \log \frac{x^2 - 1}{x}$

ANTIDERIVATIVES OF EXPONENTIAL FUNCTIONS, BASE a

1. $\int a^x dx = \int e^{(\ln a)x} dx$	2. $\int a^x dx = \frac{1}{\ln a} a^x + C$

Example 2: Find the indefinite integrals and evaluate the definite integrals.

a. $\int 3^x \, dx$

b. $\int x^2 8^{1+x^3} \, dx$

c. $\int 5^{-x}\,dx$

d. $\int_1^e \left(6^x - 2^x\right)dx$

Example 3: Find the area of the region bounded by the graph of $y = 3^{\cos x} \sin x$, $y = 0$, $x = 0$, and $x = \pi$.

Example 4: After t years, the value of a car purchased for $25,000 is $V(t) = 25,000\left(\frac{3}{4}\right)^t$.

a. Use your graphing calculator to graph the function and determine the value of the car 2 years after it was purchased.
b. Find the rates of change of V with respect to t when $t = 1$ and $t = 4$.
c. Use your graphing calculator to graph $V'(t)$ and determine the horizontal asymptote of $V'(t)$. Interpret its meaning in the context of the problem.

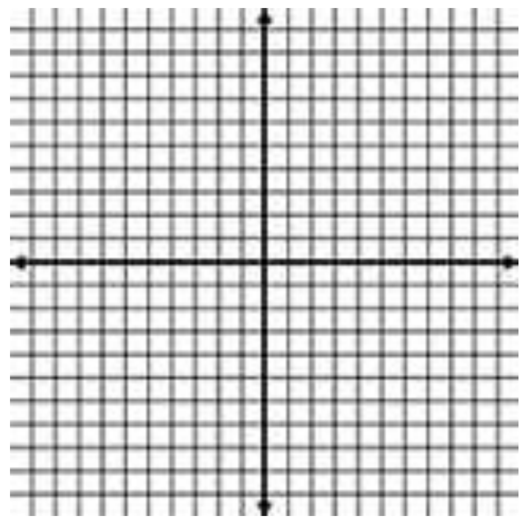
5.6: Inverse Trigonometric Functions: Differentiation

When you are done with your homework you should be able to…

- π Develop properties of the six inverse trigonometric functions
- π Differentiate an inverse trigonometric function
- π Review the basic differentiation rules for elementary functions

Warm-up: Draw the following graphs by hand from $[-\pi, \pi]$. List the domain and range in interval notation.

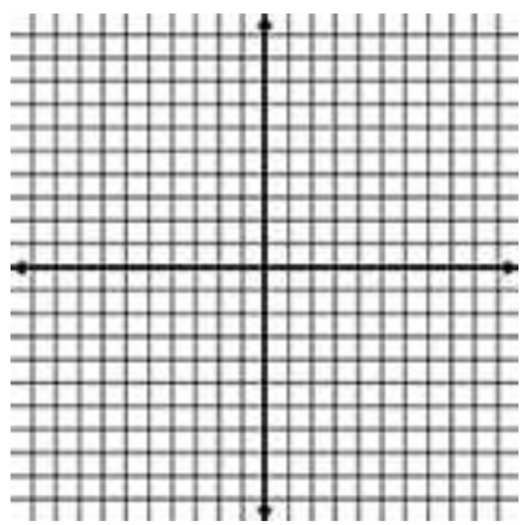
a. Graph $f(x) = \sin x$



Restricted Domain:____________

Range:______________

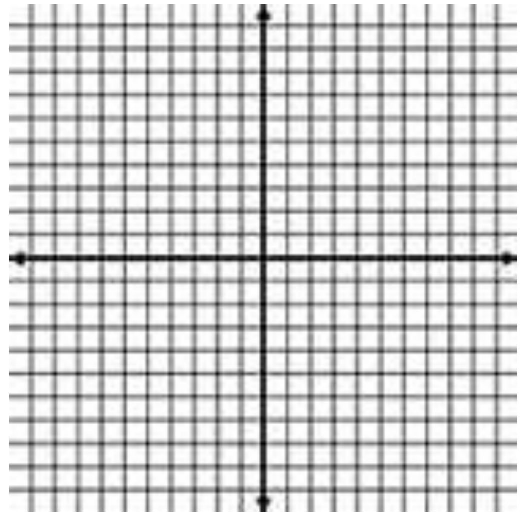
b. Graph $g(x) = \arcsin x$.



Domain:_____________

Range:______________

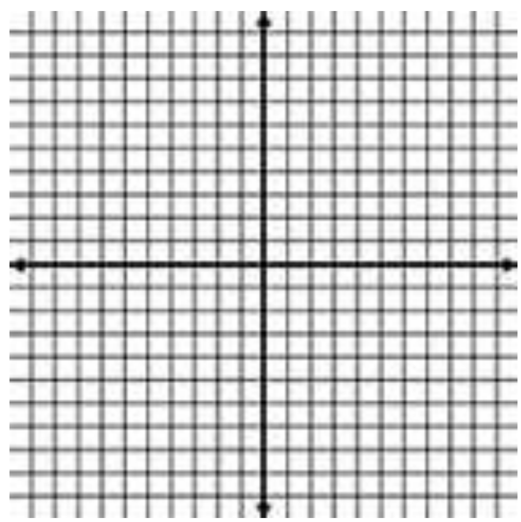
c. Graph $f(x) = \csc x$.



Restricted Domain:____________

Range:______________

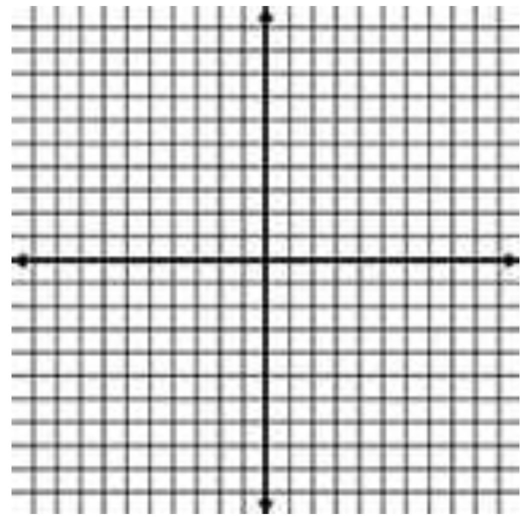
d. Graph $g(x) = \operatorname{arc}\csc x$.



Domain:_____________

Range:______________

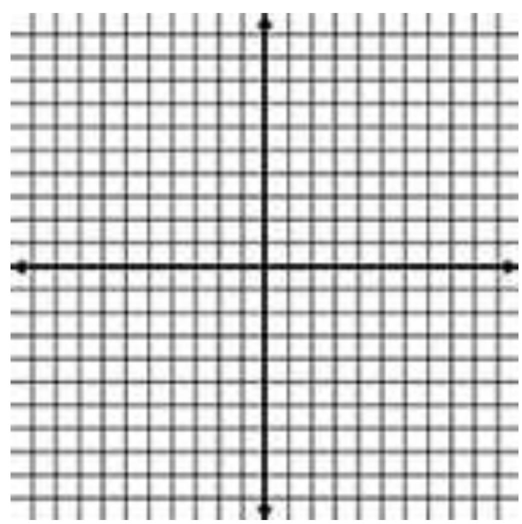
e. Graph $f(x) = \cos x$.



Restricted Domain:____________

Range:_____________

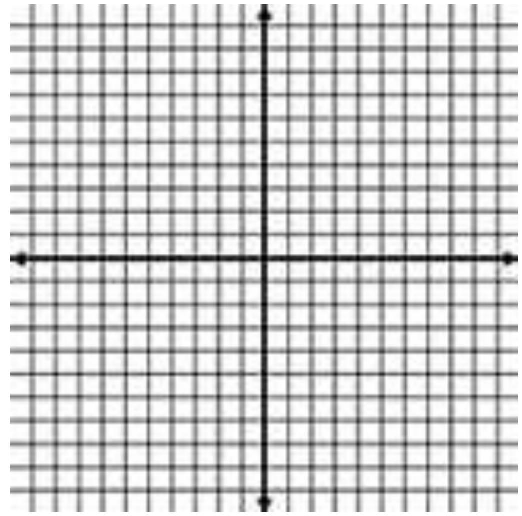
f. Graph $g(x) = \arccos x$



Domain:____________

Range:_____________

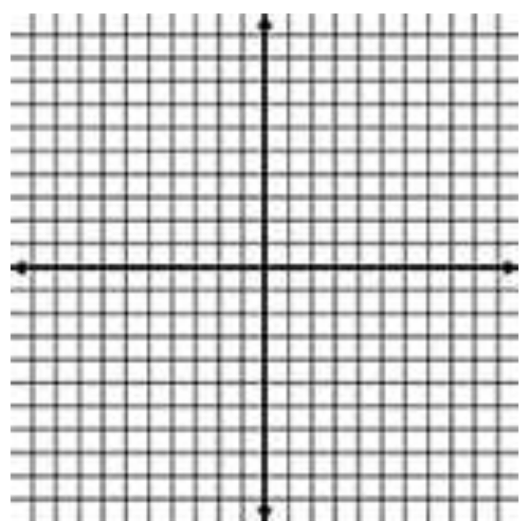
g. Graph $f(x) = \sec x$.



Restricted Domain:_____________

Range:______________

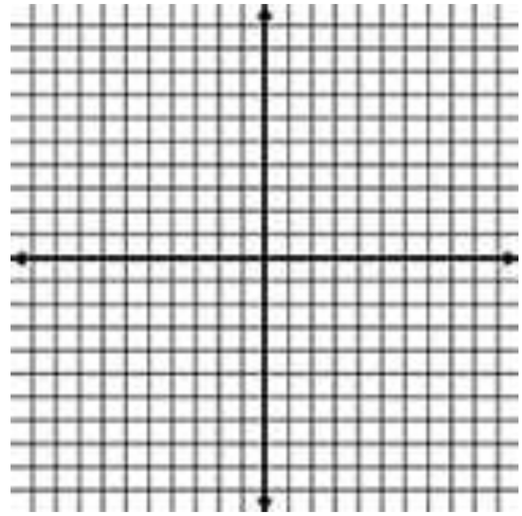
h. Graph $g(x) = \operatorname{arc}\sec x$.



Domain:_____________

Range:______________

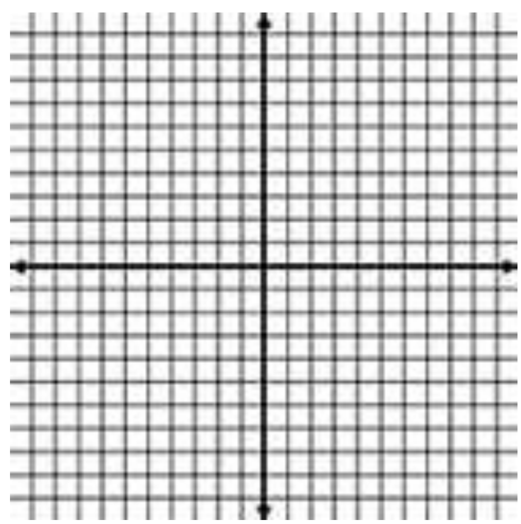
i. Graph $f(x) = \tan x$.



Restricted Domain:____________

Range:______________

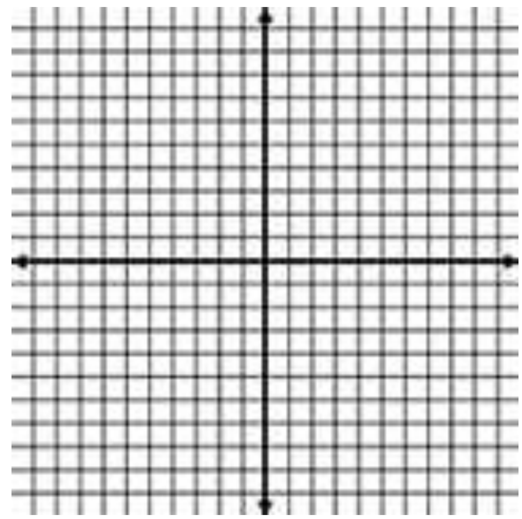
j. Graph $g(x) = \arctan x$.



Domain:_____________

Range:______________

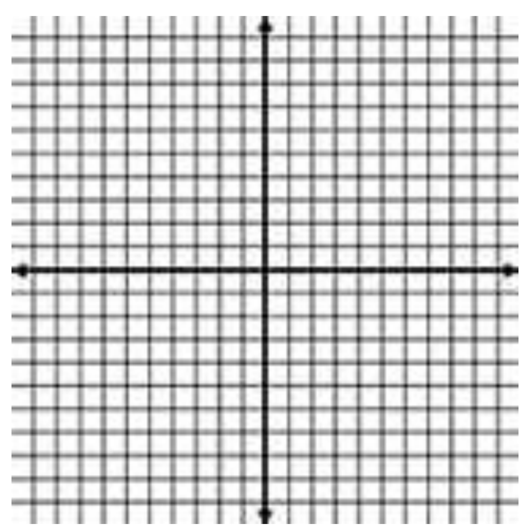
k. Graph $f(x)=\cot x$.



Restricted Domain:_____________

Range:______________

l. Graph $g(x)=\operatorname{arc}\cot x$.



Domain:_____________

Range:______________

Example 1: Evaluate each function.

a. $\operatorname{arc}\cot(1)$

b. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

c. $\operatorname{arc}\sec\left(\frac{2\sqrt{3}}{3}\right)$

d. $\arctan\left(\sqrt{3}\right)$

e. $\arccos\left(-\frac{1}{2}\right)$

f. $\operatorname{arc}\csc\left(-\sqrt{2}\right)$

Example 2: Solve the equation for *x*.

$\arctan(2x-5)=-1$

Example 3: Write the expression in algebraic form. (HINT: Sketch a right triangle)

a. $\sec\left(\arctan 4x\right)$

b. $\cos\left(\arcsin x\right)$

Example 4: Differentiate with respect to x.

a. $y = \arcsin x$

b. $y = \arccos x$

c. $y = \arctan x$

d. $y = \operatorname{arc}\csc x$

e. $y = \operatorname{arcsec} x$

f. $y = \operatorname{arc}\cot x$

What have we found out?!

DERIVATIVES OF INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x.

1. $\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
2. $\frac{d}{dx}[\arccos u] = -\frac{u'}{\sqrt{1-u^2}}$
3. $\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$
4. $\frac{d}{dx}[\operatorname{arc}\cot u] = -\frac{u'}{1+u^2}$
5. $\frac{d}{dx}[\operatorname{arc}\sec u] = \frac{u'}{|u|\sqrt{u^2-1}}$
6. $\frac{d}{dx}[\operatorname{arc}\csc u] = -\frac{u'}{|u|\sqrt{u^2-1}}$

Example 5: Find the derivative of the function. Simplify your result to a single rational expression with positive exponents.

a. $f(t) = \arcsin t^3$

b. $g\left(x\right)=\arcsin x+\arccos x$

c. $y=x\arctan 2x-\frac{1}{4}\ln\left(1+4x^2\right)$

d. $y = 25\arcsin\frac{x}{5} - x\sqrt{25-x^2}$

Example 6: Find an equation of the tangent line to the graph of the function $y = \frac{1}{2}\arccos x$ at the point $\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$.

5.7: Inverse Trigonometric Functions: Integration

When you are done with your homework you should be able to…

- π Integrate functions whose antiderivatives involve inverse trigonometric functions
- π Use the method of completing the square to integrate a function
- π Review the basic integration rules involving elementary functions

Warm-up:

1. Differentiate the following functions with respect to *x*.

a. $y = \arctan\frac{x}{2} - \frac{1}{2\left(x^2+4\right)}$

b. $\arctan(xy) = \arcsin(x+y)$.

2. Complete the square.

a. $3+4x-x^2$

b. $2x^2-6x+9$

What did you notice about the derivatives of the inverse trigonometric functions?

THEOREM: INTEGRALS INVOLVING INVERSE TRIGONOMETRIC FUNCTIONS

Let u be a differentiable function of x, and let $a>0$.

1. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C$

2. $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a}+C$

3. $\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a}+C$

Example 1: Find the indefinite integral, or evaluate the definite integral.

a. $\int \frac{dx}{x\sqrt{x^2-1}}$

b. $\int \frac{xdx}{\sqrt{x^2-1}}$

c. $\int \frac{dx}{\sqrt{1-x^2}}$

d. $\int \frac{dx}{x \ln x}$

e. $\int \frac{(\ln x)^2 \, dx}{x}$

f. $\int \ln x dx$

Example 2: Find the integral by completing the square.

a. $\int \frac{dx}{x^2+4x+13}$

b. $\int \frac{dx}{x\sqrt{x^4 - 4}}$

c. $\int \frac{2dx}{\sqrt{-x^2+4x}}$

d. $\int \frac{2x-5}{x^2+2x+2}\,dx$

e. $\int \frac{x}{\sqrt{9+8x^2-x^4}}dx$

f. $\int_1^3 \frac{1}{\sqrt{x}\left(1+x\right)}dx$

g. $\int_0^{\pi/2} \frac{\cos x}{1+\sin^2 x} dx$

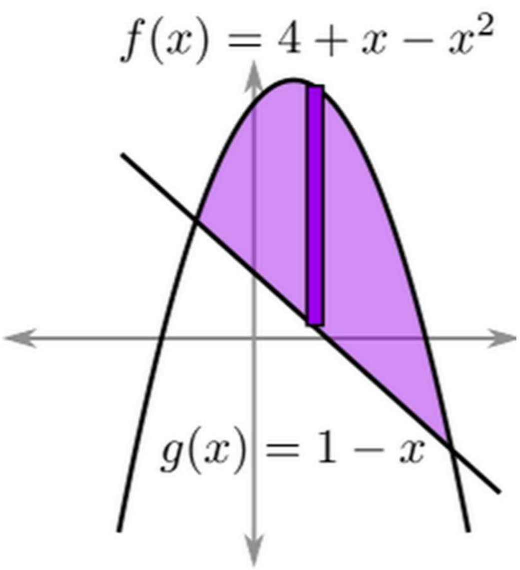
Example 3: Find the area of the region bound by the graphs of

$y = \dfrac{4e^x}{1+e^{2x}}$, $x = 0$, $y = 0$ and $x = \ln\sqrt{3}$.

7.1: Area of a Region Between Two Curves

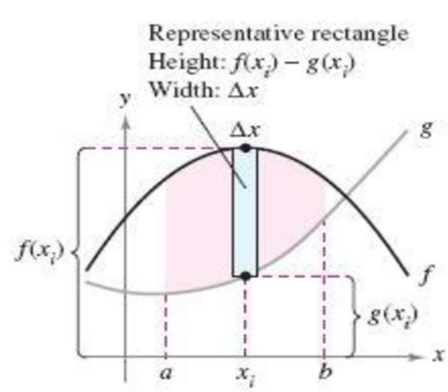
When you finish your homework you should be able to…

- π Find the area of a region between two curves using integration.
- π Find the area of a region between intersecting curves using integration.
- π Describe integration as an accumulation process.



Consider two functions that are continuous on an interval [a,b]. Also, the graph of g(x) lies below the graph of f(x). We can see that the area of the region between the graphs can be thought of as the area of the region under *g* subtracted from the area of the region under *f*.

To verify this in general, partition the interval into *n* subintervals, each of width Δx. Notice the area of the rectangle formed is $\Delta A_i = (height)(width) = [f(x_i) - g(x_i)]\Delta x$. Because *f* and *g* are both continuous on [a,b], so is *f* – *g*, and the limit exists. So…



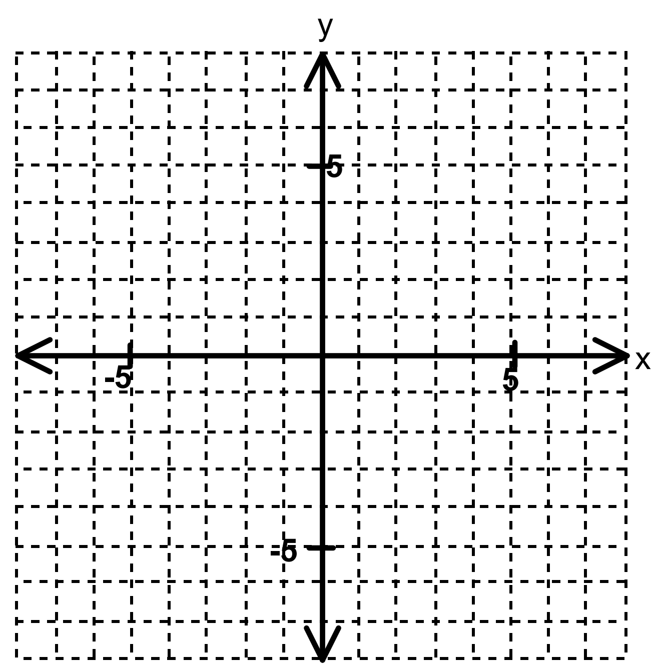
$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}[f(x_i)-g(x_i)]\Delta x = \int_a^b [f(x)-g(x)]dx$$

Area of a Region Between Two Curves

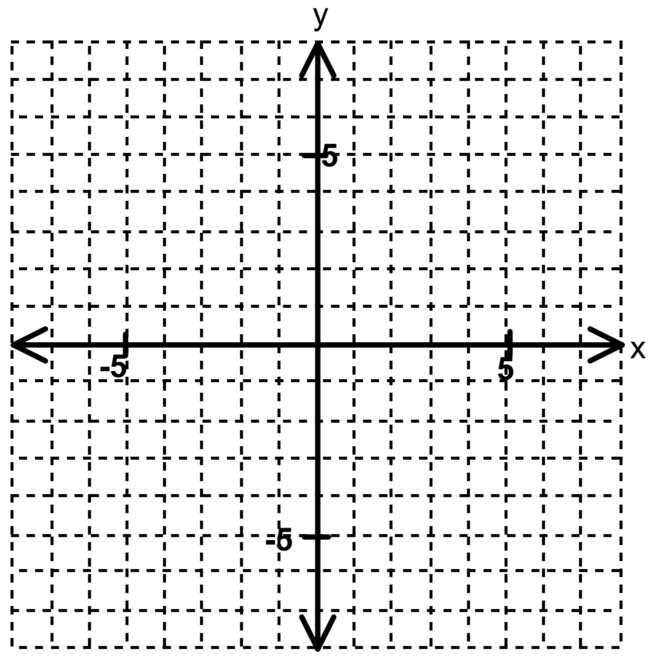
If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

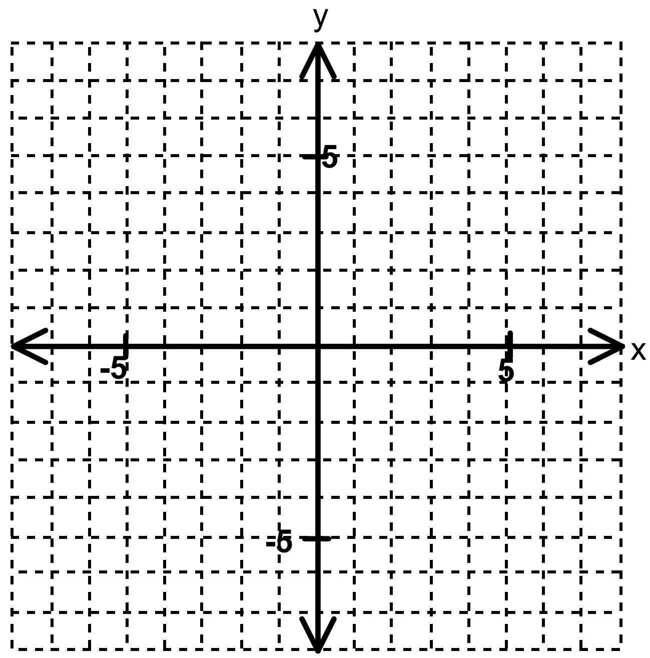
Example 1. Find the area bounded by the graphs of $y = x^3$, $y = 3x^2 - 2x$, $x = 0$, and $x = 2$.



Example 2. Sketch and find the area of the bounded region between the two curves $y = x^2$ and $y = 1 + \frac{x^2}{2}$.



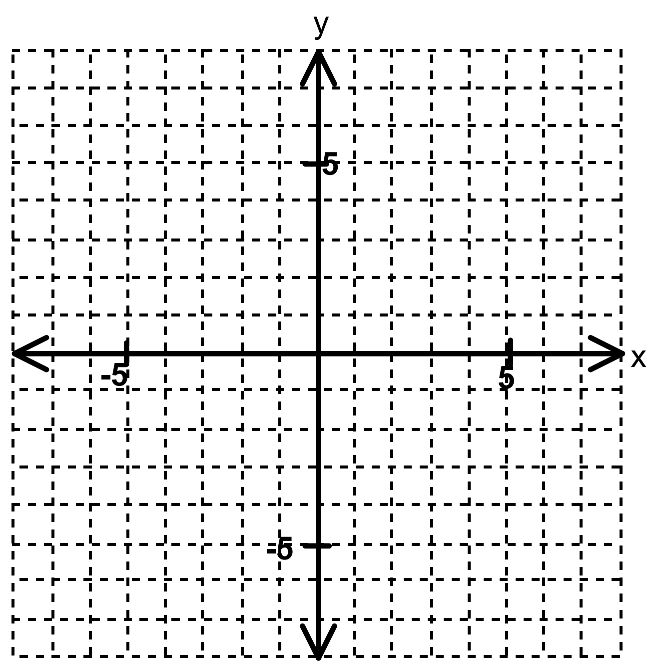
Example 3. Find the area between the graphs $x = y^2 - 2$ and $y = x$.



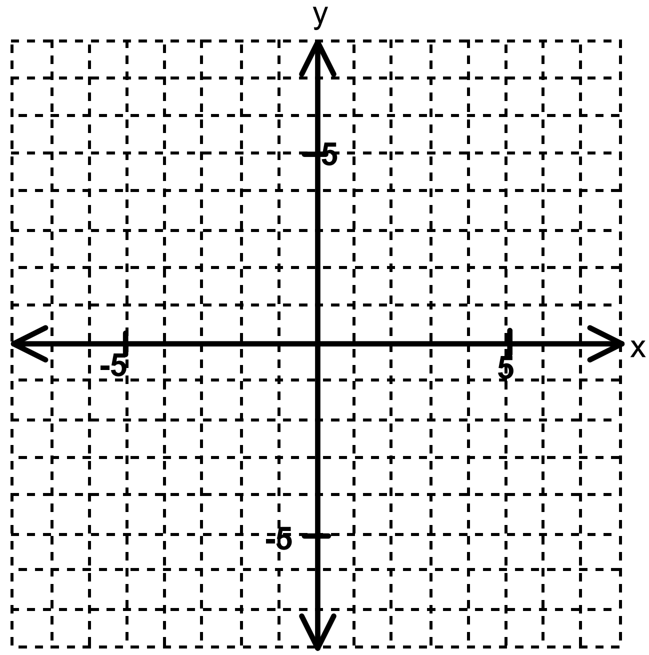
Functions of y

Now, consider the trick of regarding x as a function of y to make the calculations much simpler.

$x = y^2 - 2$ and $y = x.$



Example 4. Find the area between the graphs $f(y)=\sqrt{y+1}$ and $f(y)=y-1$.



7.2: Volume: The Disk Method

When you are done with your homework, you should be able to…

- π Find the volume of a solid of revolution using the disk method
- π Find the volume of a solid of revolution using the washer method
- π Find the volume of a solid with known cross sections

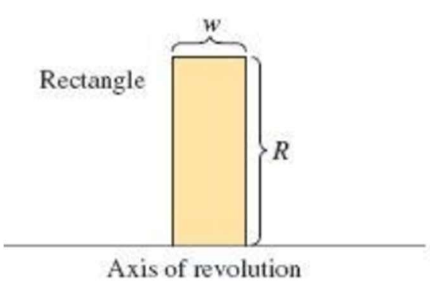
The Disk Method

When a rejoin in the plane is revolved about a line, the resulting solid is a

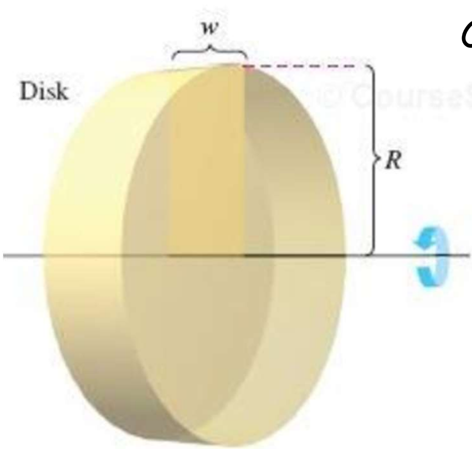
______________ _____ ___________________, and the line is called the

___________ _____ ____________________.

Examples of solids of revolution commonly seen are axles, bottles, funnels, etc. but the simplest solid of revolution is a **disk** which is formed by revolving a rectangle about an axis adjacent to one side of the rectangle.



$$\text{Volume of a disk} = (\text{area of disk})(\text{width of disk}) = \pi R^2 w$$



Consider revolving the rectangle about the axis Of revolution. When we do this, a disk is generated whose volume is:

$$\Delta V = \pi R^2 \Delta x$$

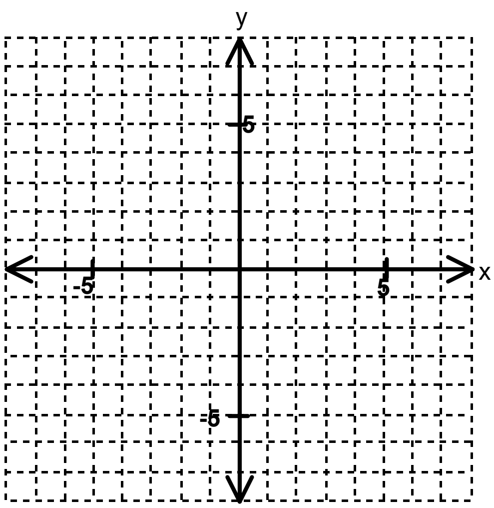
Approximating the volume of a solid by N disks of witdh Δx and radius $R(x_i)^2$ gives us

$$\sum_{i=1}^{n} \pi [R(x_i)]^2 \Delta x$$

And since the approximation gets better as $\|\Delta\| \to 0$ we can define:

$$\text{Volume of a Solid} = \lim_{\|\Delta\| \to 0} \pi \sum_{i=1}^{n} [R(x_i)]^2 \Delta x = \pi \int_a^b [R(x)]^2 \, dx$$

Example 1 The region under the graph of x^2 on [0,1] is revolved about the x axis. Sketch the resulting solid and find its volume.



We can also use the vertical axis as the axis of revolution.

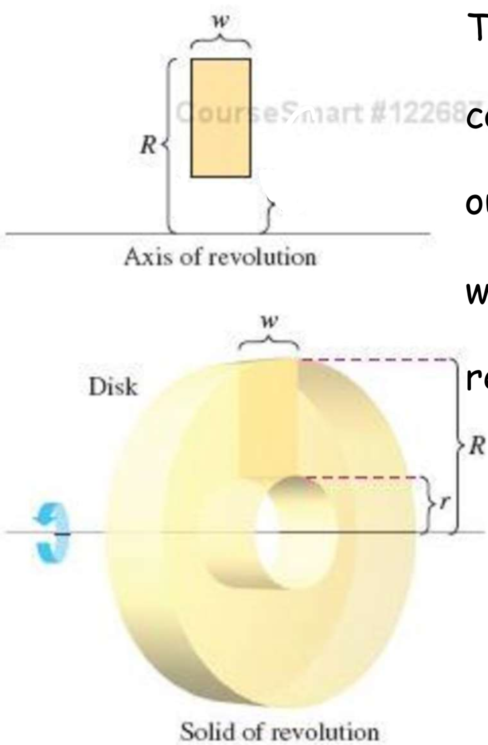
THE DISK METHOD

To find the volume of a solid of revolution with the **disk method,** use one of the formulas below. (See Figure 7.15.)

Horizontal Axis of Revolution	**Vertical Axis of Revolution**
$\text{Volume} = V = \pi \int_a^b [R(x)]^2\, dx$	$\text{Volume} = V = \pi \int_c^d [R(y)]^2\, dy$

The Washer Method

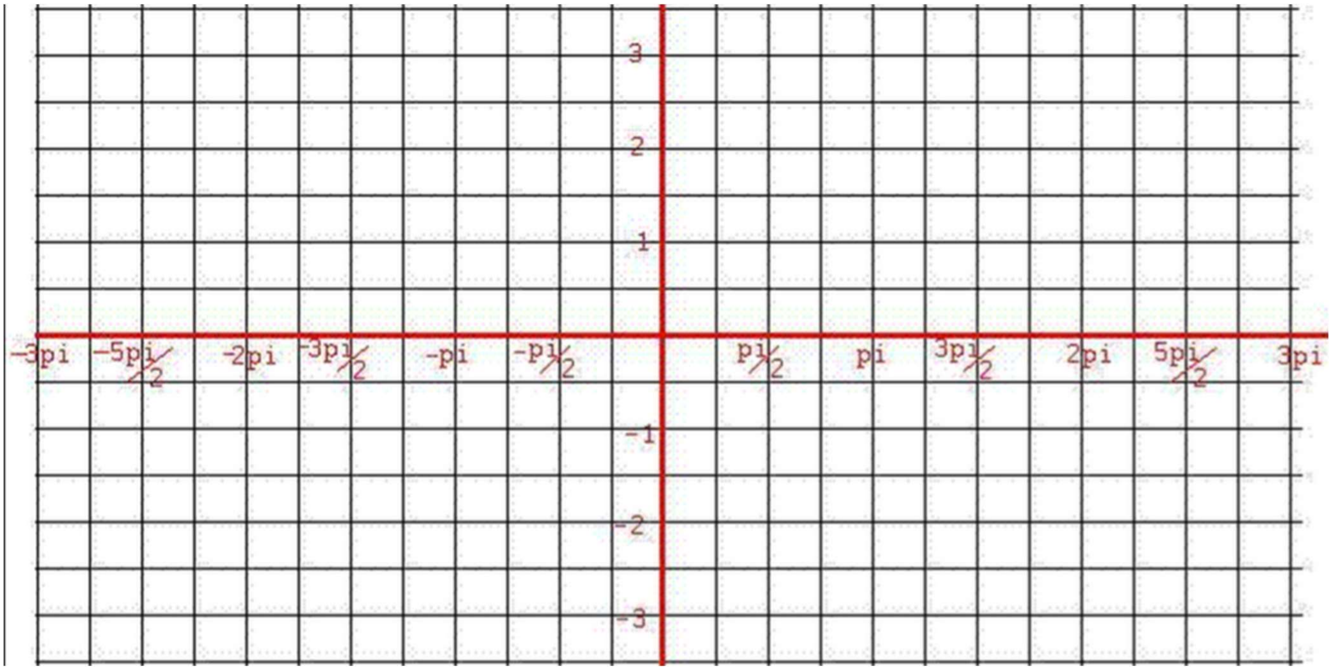
The washer method is a way of finding the volume of a solid of revolution when it has a hole in the middle. Thus named because a disk with a hole in the middle is a washer.



To see how we find the volume, we must consider the inner and outer radii. Let's call the outer radius $R(x)$ and the inner radius $r(x)$. If we revolve the region about the axis of revolution, the volume of the solid is:

$$V = \pi\int_a^b ([R(x)]^2 - [r(x)]^2)dx$$

Example 2 Sketch and then find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sin x$ and $y = x$ on [0, π/2] about the x-axis.



Example 3. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 2x^2,\ y = 0,\ x = 2$ about the a) y-axis, and then b) the line y = 8.

Solids with Known Cross Sections

We have looked at finding the volume of a solid having a circular cross section where $A = \pi R^2$ But we can generalize this method to solids of any shape as long as we know a formula for the area of an arbitrary cross section.

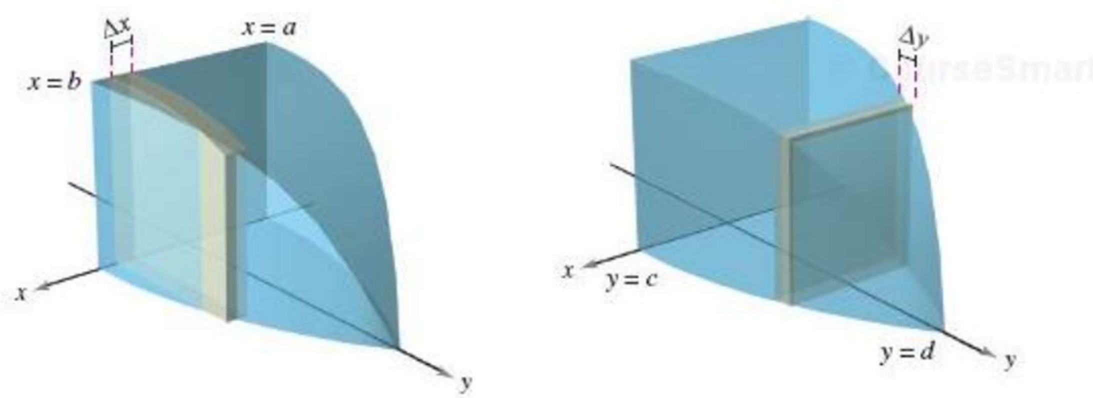
VOLUMES OF SOLIDS WITH KNOWN CROSS SECTIONS

1. For cross sections of area $A(x)$ taken perpendicular to the x-axis,

$$\text{Volume} = \int_a^b A(x)\,dx.$$ See Figure 7.24(a).

2. For cross sections of area $A(y)$ taken perpendicular to the y-axis,

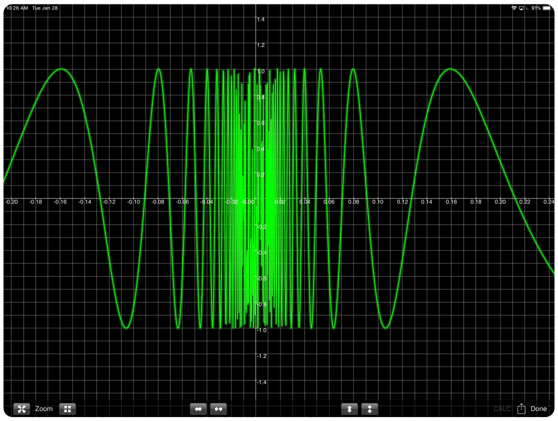
$$\text{Volume} = \int_c^d A(y)\,dy.$$ See Figure 7.24(b).



(a) Cross sections perpendicular to x-axis **(b)** Cross sections perpendicular to y-axis
Figure 7.24

Example 4. Use the disk method to verify that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$ where *r* is the radius of the base and *h* is the height.

Example 5. A manufacturer drills a hole through the center of a metal sphere of radius 6. The hole has radius r. What value of r will produce a ring whose volume is exactly half the volume of the sphere?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